A Note on Singularly Perturbed System

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Abstract
In this paper the solution of a singularly perturbed nonlinear ordinary differential equation which is obtained by using the asymptotic expansion method and a numerical method is analyzed in various values of the parameter of the equation. The numerical method which we use is Runge-Kutta-Feldbergh method of order 45, combined with the shooting method. Both solutions are verified using the first integral of the equation.

Keywords: Nonlinear Differential equations, Asymptotic Expansion,

1 Introduction
In lots of applications, a system of nonlinear ordinary differential equations depending on parameters arises naturally as a model (see for instance\(^1\)-\(^2\)). Think of a mechanical system with parameters like gravitation constant, spring constant, etc. In general, such a system as above can not be solved analytically. People then start to think about approximation technique. If the system we consider depends on a particular parameter which is small compared to the variables and other parameters, then we can use the so-called Perturbation Method to construct either an approximate solutions or an approximate system (See for instance\(^3\)). In this paper our small parameter will be named \(\varepsilon\).

Loosely speaking, perturbation method is a method of continuation of solutions of the unperturbed equation (that is when \(\varepsilon = 0\)) which is simpler than the original problem. An example of such a method are, the averaging method\(^4\) or the multi-time scale method\(^5\). The averaging method is classical; it is dated back to the time of Lagrange. Goes along in this line, a more geometrical and mathematically rigorous method of normalizations has been developed by Poincaré in the late 19th century (see\(^6\)). The multi-time scale is relatively new method developed by Kevorkian and his colaborators.

This classical notion of perturbation technique being a continuation of the unperturbed problem is, of course, no longer true in the case of singularly perturbed problem. The reason for this is since the unperturbed problem might be completely different from the original one. However, the philosophy is still the same. We assume that the solutions depend analytically on parameter \(\varepsilon\). The difference is that in singularly perturbed problems, we need to differ expansions in different part of the domain (time or spatial domain). For instance, consider

\[
\varepsilon \frac{d^2 y}{dx^2} + y^2 - y = 0.
\]

The unperturbed problem is an algebraic equation (thus there is no dynamics). Another way of constructing an approximate solution is by using a numerical method. There are lots of different ways of integration a system of differential equation (see\(^7\) for introduction to numerical method for integrating differential equations). Nevertheless, it is known that the discretization done by the numerical method, has an artefact which might lead to a wrong conclusion being read out of the output. Even more, the question of convergence in
a numerical method is also a hot issue apart from the accuracy of the results.

Our goal in this paper is to show by means of an example that the mathematical information about the system, is a very important thing to consider during numerical integrations. Knowing this information, we can decide which integration method to apply to a particular problem. For comparison in a more general problem, see\(^{59}\). In this paper consider a boundary value problem

\[ \varepsilon \frac{d^2y}{dx^2} + y \frac{dy}{dx} - y = 0, \quad 0 \leq x \leq 1 \]  

(1)

where \( 0 \ll \varepsilon \) with boundary conditions \( y(0) = A \) and \( y(1) = B \). This system is a relatively simple example of singularly perturbed problem which is actually integrable. Thus we can compare the solutions of different methods.

We start with formulating the problem in Section 2. As we mentioned, we can compute the analytic solution of (1). This solution later on will be used to check the numerical result and also the result of the perturbation method. Depending on the boundary condition given to (1), we have different cases of boundary layer in time. We expand the solution of (1) into two expansions, i.e. outer expansion and inner expansion. We then have to match this two expansions to get the asymptotic solution for the whole domain. This is done in section 3. This information is important for numerical integration. One can see that depending on the location of the layer, the numerics behave differently.

In section 4 we apply the well-known numerical integrations scheme for boundary value problem: shooting method. To be more precise, we apply a simple shooting method, which consists of one point shooting. The correction of the shooting method is achieved by bisection method. One may see that this is clearly not optimal in any sense. However, for our purpose this is good enough. We study the convergence of the shooting method as the parameter \( \varepsilon \) decreases to zero. what we mean by this is we numerically show an example how the domain of convergence shrink as \( \varepsilon \to 0 \). Also, we show numerically that the location of the boundary layer is rather important to improve the numerics.

2. First integral

It is instructive to write equation (1) as a system of first order differential equations

\[ \frac{du_1}{dx} = u_2 \]  

\[ \frac{du_2}{dx} = \frac{1}{\varepsilon} u_1 (1-u_2) \]  

(2)

where \( u_1(x) = y(x) \) and \( u_2(x) = \frac{dy}{dx} u_2(x) \).

Consider a function: \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by

\[ F(u_1,u_2) = \frac{1}{\varepsilon} u_1^2 + \varepsilon (u_2 + \ln |u_2 - 1|). \]  

(3)

We can calculate the derivative of this function along the solution of (2), i.e.

\[ \frac{\partial F}{\partial u_1} \frac{du_1}{dx} + \frac{\partial F}{\partial u_2} \frac{du_2}{dx} = 0. \]  

Since \( F(u_1,u_2) \) is a nonconstant real-valued \( C^1 \) function on any open subset of \( \mathbb{R} \setminus \{u_2 = 1\} \) then \( F(u_1,u_2) \) is a first integral of the system (2). This statement is valid for a general \( n \)-dimensional system of differential equations \( \xi' = v(\xi) \) (where the prime denotes derivative with respect with the independent variables \( x \in \mathbb{R} \)). A non constant \( C^1 \) function \( F \) define on a open set of the phase-space, is called an integral of the system \( \xi' = v(\xi) \), if the derivative of \( F \) along the orbit of the system vanishes everywhere.

Consider a level set of the integral:

\[ F^{-1}(c) = \{ \xi \mid F(\xi) = c \}. \]  

This is a co-dimension one \( C^1 \)-manifold if the spatial derivative \( D_\xi F \neq 0 \). On the other hand, an integral by definition is a function which is kept constant along the orbit of the system. Thus, if \( \xi(t) \) is a solution of \( \xi' = v(\xi) \) and \( \xi(0) \in F^{-1}(c) \) for a \( c \in \mathbb{R} \), then we have \( \xi(t) \in F^{-1}(c) \) for all \( t \in \mathbb{R} \). The value \( c \) is called a regular value of \( F \).

Let \( F_j \), \( j = 1, \ldots, m \) be \( C^1 \) integrals of the system \( \xi' = v(\xi) \). Consider a function \( F = (F_1, \ldots, F_m) \) which is \( C^1 \) (\( r \) is the minimum of \( r_j, j = 1, \ldots, m \)) function from \( \mathbb{R}^n \) to \( \mathbb{R}^m \). Let \( e = (e_1, \ldots, e_m) \) where \( e_j \) is a regular value of \( F_j \). If \( D_\xi F(e) \) has maximal rank, then the level set \( F^{-1}(e) \) defined a \( C^1 \) codimension \( m \) manifold in \( \mathbb{R}^n \). Thus, if \( m = n - l \), we have \( F^{-1}(e) \) is a codimension \( n - 1 \) or simply a one dimensional manifold. This manifold is the intersection of the manifolds \( F_j^{-1}(c_j) \). By parametrizing this one dimensional manifold \( F_j^{-1}(c_j) \), we derive the solution of \( \xi' = v(\xi) \). The value \( e \) can be chosen to satisfy the initial condition \( \xi(0) = \xi_0 \). More details on geometrical methods in dynamical systems can be found in\(^{60}\).

Our problem in a sense much easier, since we are dealing with a planar system. Thus we need only one integral and the level set of that
integral immediately gives us the solution. The problem is only to specify the regular value $c$. However, in a boundary value problem as it is in our problem, the determination of $c$ to satisfy the boundary condition is not easy. This first integral (3) will be used as a verifying tool for solution of equation (1) which is obtained by the asymptotic expansion method and the numerical method in our work later on. The value $c$ will be calculated also numerically.

3. Asymptotic expansion method

To obtain the solution of equation (1) by asymptotic expansion method, we must find first the outer expansion and the inner expansion of $y(x)$, and then match the expansions. The outer expansion is useful to approximate $y(x)$ outside of the boundary layer region, and the inner expansion is useful to approximate $y(x)$ inside of the boundary layer region.

3.1 Outer Expansion

Let us fix $\varepsilon \ll 1$. We assume that the solution $y(x)$ can be expanded in $\varepsilon$, i.e.

$$y(x) = h(x) + \varepsilon h_1(x) + \varepsilon^2 h_2(x) + \ldots.$$  \hfill (4)

By substituting the outer expansion (4) to equation (1) we get

$$h'(\frac{dh}{dx}) - h = 0$$ \hfill (6)

with boundary conditions: $h(0) = A$ and $h(1) = B$.

Equation (6) has solutions

$$h(x) = 0 \text{ or } h(x) = x + k, \ k \in 3.$$ \hfill (7)

Clearly, $h(x) = 0$ in general can not satisfy the boundary condition. It is also important to realize that both solutions in (7) are solving the equation (1) exactly. It then implies that $h_j(x) = 0, j = 1, 2, \ldots$.

The integration constant $k$ need to be chosen to satisfy the boundary condition. This is clearly an overdetermined problem which is in general, has no solution. In fact it has solution only in the case where $B - A = 1$. If $A \neq B - 1$, equation (1) can produce a boundary layer at $x = 0$ or $x = 1$ or a shock layer near a point in open interval $(0, 1)$. In this paper we will restrict our discussion in the boundary layer cases.

Suppose that the boundary layer occurs at the left ($x = 0$), then equation (7) will satisfy the right boundary condition ($h(1) = B$). Therefore the outer expansion of $y(x)$ becomes $h_j(x) = x - 1 + B$ and we get that $h_j(0) = B - 1$, where the subscript $R$ denotes that the right boundary condition is satisfied. If the boundary layer occurs at the right ($x = 1$), then equation (7) will satisfy the left boundary condition ($h(0) = A$). Therefore the outer expansion of $y(x)$ becomes $h_j(x) = x - A$ and we get that $h_j(x) = A + 1$, where the subscript $L$ denotes that the left boundary condition is satisfied.

3.2. Inner expansion

Let us make a stretched variable $x' = (x - x_0)/\delta_\varepsilon$ where $x_0$ is a point where a boundary layer occurs and $\delta_\varepsilon$ is an order function (for definition of order function see(10)). In general we can choose any order function but for simplicity, think of the family $\varepsilon^\nu, \nu > 0$. This order function indicates how large the boundary layer is. The stretchal variable has an effect of magnifying the boundary layer region and thereby eliminate any rapid variation that might be exhibited by $y(x)$. It can be viewed as a scaling procedure to smoothen out the solution inside the boundary layer.

Let us come back to the system (2). In the stretched variables the system look like (after rescalling time)

$$\frac{du_1}{dx} = \delta_\varepsilon u_2$$
$$\frac{du_2}{dx} = \frac{\delta_\varepsilon}{\varepsilon} u_1(1 - u_2)$$ \hfill (8)

If we choose $0 < \nu < 1$, we see that we will end up with a system having the equivalent asymptotic ordering as in (2). Also if we choose $\nu \geq 1$, the resulting equations are equivalent to each other. Thus, we choose $\nu = 1$.

**Remark 1.** The method of finding the order function $\delta_\varepsilon$ is known as a method of significant degeneration. This is a formal method which means that it is not acceptable as a proof of the size of the boundary layer. To derive such a size, one needs to go into the details and doing estimation for the asymptotics.

Let the inner expansion of $y(x')$ be

$$y(x') = g(x') + \varepsilon g_1(x') + \ldots$$ \hfill (9)

Substituting the inner expansion into equation (1) we obtain (we have ommited the asterix)
\[
\frac{1}{\varepsilon} \left( \frac{d^2 g}{dx^2} + g \frac{dg}{dx} \right) + \left( \frac{d g}{dx} + g \frac{dg}{dx} \right) + \varepsilon \cdots = 0. \tag{10}
\]

The \((\varepsilon^1)\) term in the equation (10) is

\[
\frac{d^2 g}{dx^2} + g \frac{dg}{dx} = 0 \tag{11}
\]

and has solutions

\[
g(x) = \beta \tanh \left( \frac{\beta}{2} (x + c) \right) \quad \text{or} \quad g(x) = \beta \coth \left( \frac{\beta}{2} (x + c) \right) \tag{12}
\]

where \(\beta\) and \(c\) are integration constants.

Figure 1. Sketch of the coth curve (left) and the tanh curve (right).

The sketch of the solutions in (12) are given in Figure 1. The tanh solution increases to its asymptotic value \(\beta\) as \(x \to \infty\) and decreases to \(-\beta\) as \(x \to -\infty\). Thus a sharp transition is expected to happen in the derivative of this solution. This is happen in the case of the occurrence of a shock layer inside the interval (0, 1). On the other hand, the coth solution decreases from infinity at \(x^* = -k\) to its asymptotic value \(-\beta\) as \(x^* \to \infty\) and increases from \(-\infty\) to its asymptotic value \(-\beta\) as \(x^* \to -\infty\). The variation in the derivative is monotonic (although also large). This coth solution can be used as an inner expansion if equation (1) has boundary layer.

Now we have constructed an asymptotic solution for (1) for both the inner and outer domain. Now we need to satisfy the boundary conditions and also to match these two solutions.

4. Matching the outer expansion with inner expansion

In this paper we constructed only an \((\varepsilon)\) approximation of the solution of (1). Thus we need only to match \(h(x)\) and \(g(x)\) together. Recall that the approximation in the inner domain is necessary since the approximate solution in the outer domain can not satisfy both of the boundary conditions simultaneously. We will first describe how to do the matching in general.

4.1 Left Boundary layer

Suppose the boundary layer occurs at the left \((x_0 = 0)\), thus for the outer expansion we use function \(h_R(x)\). To do the matching, first make a matching variable \(x_\eta = \frac{x}{\eta \varepsilon}\), where \(\varepsilon << \eta << 1\).

1. The condition to match the outer expansion and the inner expansion is

\[
\lim_{\varepsilon \to 0}^{x_\eta, \text{fixed}} h_R(x) - g(x^*) = 0 \tag{13}
\]

or in other words as

\[
\lim_{\varepsilon \to 0}^{x_\eta, \text{fixed}} h_R(\eta x_\eta) - g \left( \frac{\eta x_\eta}{\varepsilon} \right) = 0 \tag{14}
\]

Therefore the condition for the matching can be simplified into \(h_R(0) = g(\infty)\). And we obtain that the common part of this matching is \(h_R(0)\). Common parts are terms which are matched (cancel out in the matching).

So if the boundary layer occurs at the left, the solution of equation (1) using the asymptotic expansion method is

\[
y(x) = h_R(x) + g(x^*) - h_R(0) \tag{15}
\]

4.2 Right Boundary Layer

If the boundary layer occurs at the right \((x_d = 1)\), then we use function \(h_L(x)\) for the outer expansion. The matching variable is \(x_\eta = \frac{1-x}{\eta \varepsilon}\), where \(\varepsilon << \eta << 1\), and the condition to enable the matching is

\[
\lim_{\varepsilon \to 0}^{x_\eta, \text{fixed}} h_L(x) - g(x^*) = 0 \tag{16}
\]
or in other words as
\[ \lim_{x \to x_0} h_L(x) - g\left( -\frac{\eta y_n}{\varepsilon} \right) = 0 \] (17)

Thus the matching condition can be simplified into \( h_L(1) = g(\infty) \). And we obtain the common part for this matching as \( h_L(1) \).

So if the boundary layer occurs at the right, the solution of equation (1) using the asymptotic expansion method is
\[ y(x) = h_L(x) + g(x^*) - h_L(0) \] (18)

**4.3 Case 1:** \( B = A + 1 \)

As we have seen in subsection 3.1, at line \( B = A + 1 \) equation (1) is not a boundary layer problem and it has an exact solution, that is \( y(x) = h_L(x) = h_L(x) = x + A = x + B - 1 \).

**4.4 Case 2a:** \( A > B - 1 \) and \( B > 1 \)

At this region, the boundary layer occurs at the left \((x_0 = 0)\). It follows that the outer expansion of \( y(x) \) is \( y(x) = g_L(x) = x - 1 + B \), and the inner expansion is
\[ y(x, \varepsilon) = g_L(x^*) = \beta \coth \left( \frac{\beta (x^* + c)}{2} \right) \] (19)
where \( x^* = \frac{x}{\varepsilon} \); the constant integration \( \beta = B - 1 \) obtained from the matching condition \( h_L(0) = g_L(\infty) \) and the constant integration \( c = 2\beta^{-1} \coth^{-1}(A\beta^{-1}) \) obtained from the boundary condition \( y(0) = g_L(0) = A \), where the subscript \( L \) at function \( g_L(x^*) \) denotes the left boundary layer, and \( \coth^{-1} \) denotes arccoth.

The common part of this matching in this case is \( h_L(0) = B - 1 \), and according to equation (15), the solution of equation (1) using the asymptotic expansion method is
\[ y(x) = h_L(x) + g_L(x^*) - B + 1 \] (20)

Note that we use the coth solution in \( g_L(x) \) because in the closed interval \([0,1]\) it is a decreasing function and approaches the outer expansion from the left side \((x = 0)\).

In this region it is not possible that a boundary layer occurs at the right \((x_0 = 1)\), because in the closed interval \([0,1]\) there is no inner expansion which increases and approaches the outer expansion \( h_L(x) = x + A \).

**4.5 Case 2b:** \( A > B - 1 \) and \( A < -1 \)

The \( AB \) region in this case is a reflection region of \( AB \) region at Case 2a above with respect to line \( B = -A \). In this region the boundary layer occurs at the right side \((x_0 = 1)\), so the outer expansion of \( y(x) \) is \( y(x) = h_L(x) = x + A \), and the inner expansion is
\[ y(x, \varepsilon) = g_R(x^*) = \frac{\varepsilon}{\beta} \coth \left( \frac{\beta (x^* + c)}{2} \right) \] (21)
where \( x^* = \frac{x-1}{\varepsilon} \); the constant integration \( \beta = A + 1 \) obtained from the matching condition \( h_L(1) = g_R(\infty) \) and the constant integration \( c = 2\beta^{-1} \coth^{-1}(A\beta^{-1}) \) obtained from the boundary condition \( y(1) = g_R(1) = B \). The subscript \( R \) at function \( g_R(x) \) denotes the right boundary layer.

The common part for this matching is \( h_L(1) = A + 1 \), and according to equation (18) the solution of equation (1) is
\[ y(x) = h_L(x) + g_R(x^*) - A - 1 \] (22)

Note that we use the coth solution in \( g_R(x) \) because in the closed interval \([0,1]\) it is an increasing function and approaches the outer expansion from the right side.

In this region it is not possible that a boundary layer occurs at the left \((x_0 = 0)\), because in the closed interval \([0,1]\) there is no inner expansion which decreases and approaches the outer expansion \( h_R(x) = x - 1 + B \) from the left side.

**5. Numerical method**

In obtaining a solution of equation (1) numerically, we use MATLAB Version 5.3 (Release 11) in a computer with Pentium III processor. We use the built-in ode45 function in MATLAB combined with the shooting method. In this paper, we consider the left boundary case numerically with two directions of shooting, from the left side \((\eta = 0)\), where the boundary layer occurs and from the right side \((x = 1)\). The parameters are

(i) the boundary conditions are \( y(0) = A = 2.5 \) and \( y(1) = B = 2 \)
(ii) number of partitions in the closed interval \([0,1]\) is 1000
(iii) the relative tolerance and the absolute tolerance in ode45 function are set as \( 10^{-13} \)
(iv) the stopping conditions are \( |\theta - \mu| > \delta \) and \( \lambda = \sigma \), where \( \theta \) is the result of upper shooting, \( \mu \) is the result of lower shooting, \( \delta \) is an error constant in iterations of shooting method, \( \lambda \) is the slope of middle shooting at one iteration, and \( \sigma \) is the slope of middle shooting at one iteration before. If one of the two stopping conditions is reached, then the shooting method will be stopped.

(v) the error constant \( \delta \) is set as \( 10^{-10} \).
The idea behind the second stopping condition is to stop the iterations of shooting method if the computer is unable to refine the slope of shooting. If this stopping condition is reached first, then the result of numerical method does not satisfy what we want. This stopping condition is very useful when $\varepsilon$ is very small.

Figure 2 draws number of iterations to obtain the solution of equation (1) numerically with respect to parameter $\varepsilon$ and directions of the shooting. The shooting method from the left side converges for various values of parameter $\varepsilon$. Unfortunately the shooting method from the right side toward the location of boundary layer diverge when $\varepsilon < 0.085$. At these values of $\varepsilon$, the numerical method can not refine the slope of shooting anymore although the computation is done in double precision. The numerical method is stopped by the second stopping condition.

Figure 2. Number of iterations vs epsilon

Table 1 lists various initial slope of shooting with respect to the directions of the shooting and the values of parameter $\varepsilon$. The second column and third column list the initial upper slope (IUS) shooting and the initial lower slope (ILS), both for shooting from the left side (the initial position is $y(0) = A = 2.5$). The fourth column and the fifth column list the initial upper slope and the initial lower slope, both for shooting from the right side (the initial position is $y(1) = B = 2$).

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>IUS : $\frac{dy}{dx}(0)$</th>
<th>ILS : $\frac{dy}{dx}(0)$</th>
<th>IUS : $\frac{dy}{dx}(1)$</th>
<th>ILS : $\frac{dy}{dx}(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-25</td>
<td>-28</td>
<td>0.999996</td>
<td>0.9999975</td>
</tr>
<tr>
<td>0.09</td>
<td>-29</td>
<td>-30</td>
<td>0.999999</td>
<td>0.9999995</td>
</tr>
<tr>
<td>0.085</td>
<td>-30</td>
<td>-32</td>
<td>0.9999997</td>
<td>0.99999998</td>
</tr>
<tr>
<td>0.08</td>
<td>-32</td>
<td>-33</td>
<td>0.99999991</td>
<td>0.99999992</td>
</tr>
<tr>
<td>0.075</td>
<td>-33</td>
<td>-38</td>
<td>0.99999997</td>
<td>0.99999998</td>
</tr>
<tr>
<td>0.07</td>
<td>-37</td>
<td>-39</td>
<td>0.999999991</td>
<td>0.999999995</td>
</tr>
<tr>
<td>0.06</td>
<td>-43</td>
<td>-47</td>
<td>0.999999997</td>
<td>0.9999999998</td>
</tr>
<tr>
<td>0.05</td>
<td>-50</td>
<td>-55</td>
<td>0.99999999998</td>
<td>0.99999999999</td>
</tr>
<tr>
<td>0.025</td>
<td>-100</td>
<td>-110</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>0.0125</td>
<td>-200</td>
<td>-220</td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>

From Table 1 we see that the initial slope of shooting method from the right side are very sensitive to the decreasing value of parameter $\varepsilon$. When $\varepsilon = 0.025$ and $\varepsilon = 0.0125$, we can not obtain the initial slope for the upper shooting and lower shooting from the right side although the computation is done in double precision.

Hence information about the location of boundary layer is very useful for us to determine what direction of the shooting we should use in order to obtain a converge solution numerically.

6. Comparison of solutions

In this section we will compare the error of the solution which is obtained by the shooting method (from the left side) and the error of the solution which is obtained by the asymptotic expansion method, both with respect to the first integral (3).

Table 2 lists those error in various values of $\varepsilon$. The second column lists the error of solution obtained by the asymptotic expansion method. The error scheme is the maximum absolute value of the ratio of the solution obtained by the
asymptotic method minus the exact solution (obtained via the first integral (3)) divided by the exact solution. Meanwhile the third column lists the error of solution obtained by the numerical method. The error scheme is the maximum absolute value of the ratio of the solution obtained by the numerical method minus the exact solution divided by the exact solution.

Table 2. Error of solution (Asy Exp Met) Error of solution (Num Met)

<table>
<thead>
<tr>
<th>ε</th>
<th>Error of solution (Asy Exp Met)</th>
<th>Error of solution (Num Met)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$2.903406 \cdot 10^{-2}$</td>
<td>$8.801348 \cdot 10^{-11}$</td>
</tr>
<tr>
<td>0.09</td>
<td>$2.673858 \cdot 10^{-2}$</td>
<td>$3.728682 \cdot 10^{-10}$</td>
</tr>
<tr>
<td>0.085</td>
<td>$2.555283 \cdot 10^{-2}$</td>
<td>$9.190690 \cdot 10^{-10}$</td>
</tr>
<tr>
<td>0.08</td>
<td>$2.434022 \cdot 10^{-2}$</td>
<td>$3.843666 \cdot 10^{-9}$</td>
</tr>
<tr>
<td>0.075</td>
<td>$2.309945 \cdot 10^{-2}$</td>
<td>$7.417037 \cdot 10^{-9}$</td>
</tr>
<tr>
<td>0.07</td>
<td>$2.182970 \cdot 10^{-2}$</td>
<td>$2.472208 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>0.06</td>
<td>$1.919711 \cdot 10^{-2}$</td>
<td>$6.298429 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>0.05</td>
<td>$1.643030 \cdot 10^{-2}$</td>
<td>$5.497683 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>0.025</td>
<td>$2.915364 \cdot 10^{-4}$</td>
<td>$2.915364 \cdot 10^{-1}$</td>
</tr>
<tr>
<td>0.0125</td>
<td>$4.039073 \cdot 10^{-1}$</td>
<td>$4.039073 \cdot 10^{-1}$</td>
</tr>
</tbody>
</table>

We see that when the values of ε decreases, the error of the solution obtained by the numerical method becomes bigger and bigger. When ε = 0.0125 the error of the solution obtained by the numerical method has been in same magnitude with the error of the solution obtained by the asymptotic expansion method.

Hence the solution obtained by the asymptotic expansion method becomes prominent when the value of ε is very small.

7. Conclusion

We see that equation (1) can be a boundary layer problem or not depending on the value of its boundary condition. Information about the location of boundary layer problem can help us to determine the direction of shooting method in order to get a converge solution numerically. This information is very useful when the value of ε becomes smaller and smaller. Numerical method has some disadvantages, especially when the value of ε is very small. For the value of ε is very small the solution obtained by the asymptotic method becomes prominent.

References