Fundamental of Control Systems
– Steady State Error

Lecturer:
Dr. Wahidin Wahab M.Sc.
Aries Subiantoro, ST. MSc.

Electrical Engineering Department
University of Indonesia
Steady State Error

- How well can a system track a standard input
  - step, ramp, parabola
  - based on final value theorem of Laplace Transforms

Steady state value of $e(t)$ is

$$e(\infty) = \lim_{t \to \infty} \{e(t)\} = \lim_{s \to 0} \{sE(s)\}$$
Table 7.1
Test waveforms for evaluating steady-state errors of position control systems

<table>
<thead>
<tr>
<th>Waveform</th>
<th>Name</th>
<th>Physical interpretation</th>
<th>Time function</th>
<th>Laplace transform</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Step</td>
<td>Constant position</td>
<td>1</td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td></td>
<td>Ramp</td>
<td>Constant velocity</td>
<td>( t )</td>
<td>( \frac{1}{s^2} )</td>
</tr>
<tr>
<td></td>
<td>Parabola</td>
<td>Constant acceleration</td>
<td>( \frac{1}{2}t^2 )</td>
<td>( \frac{1}{s^3} )</td>
</tr>
</tbody>
</table>
Figure 7.2
Steady-state error:
\(a\). step input;
\(b\). ramp input
Steady State Error

Consider a unity feedback system

- note that any closed loop system may be reduced to a unity gain system
Steady State Error

\[ R(s) \xrightarrow{+} E(s) \xrightarrow{-} \begin{array}{c} \bigcirc \\ + \\ - \end{array} \xrightarrow{G(s)} C(s) \]

Steady state error \( e_{ss} = \lim_{s \to 0} \{sE(s)\} \)

\[ \frac{C(s)}{R(s)} = \frac{E(s) + R(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \]

\[ \frac{E(s)}{R(s)} = \frac{G(s)}{1 + G(s)} - 1 = \frac{1}{1 + G(s)} \]
Steady State Error

\[
\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} \rightarrow E(s) = R(s) \cdot \frac{1}{1 + G(s)}
\]

\[
\therefore e_{ss} = \lim_{s \to 0} \left\{ \frac{sR(s)}{1 + G(s)} \right\}
\]
Error Constants

- Error constants give figure of merit for steady state error
- The bigger the error constant, the better the steady state error
- Tracking of input signal defined in terms of response to standard inputs
  - step (position), ramp (velocity), parabola (acceleration)
System types

- Systems classified into how many open loop poles lie at the origin
- e.g. type 0 open loop systems have no poles at the origin, type 1 systems have one pole at the origin, and so on
The steady state error of a unity feedback system for a unit step input is

\[ e_{ss} = \lim_{s \to 0} \left\{ s \cdot \frac{1}{1 + G(s)} \cdot \frac{1}{s} \right\} = \frac{1}{1 + G(0)} \]

Define static position error constant

\[ K_p = \lim_{s \to 0} \{ G(s) \} = G(0) \]
Static Position Error Constant

\[ e_{ss} = \frac{1}{1 + K_p} \]

Note for a type 0 system \( K_p \) is finite, \( e_{ss} \) is finite
for a type 1 or higher system it is infinite
\( e_{ss} \) is zero

If zero steady state error is required for a step input
a first or higher order system is needed.
Static Velocity Error Constant

Steady state response for system to unit ramp input

\[ e_{ss} = \lim_{s \to 0} \left\{ s \cdot \frac{1}{1 + G(s)} \cdot \frac{1}{s^2} \right\} = \lim_{s \to 0} \left\{ \frac{1}{sG(s)} \right\} \]

Define static velocity error constant

\[ K_v = \lim_{s \to 0} \{sG(s)\} \]

\[ \rightarrow e_{ss} = \frac{1}{K_v} \]
Static Velocity Error Constant

For a type 0 system $K_v = 0 \rightarrow e_{ss}$ is infinite

Type 1 system $K_v$ is finite $\rightarrow e_{ss}$ is finite

Type 2 or higher system $K_v$ is infinite $\rightarrow e_{ss}$ is zero
Static Acceleration Error

Constant

Steady state response for system to unit parabola input

\[ e_{ss} = \lim_{s \to 0} \left\{ s \cdot \frac{1}{1 + G(s)} \cdot \frac{1}{s^3} \right\} = \lim_{s \to 0} \left\{ \frac{1}{s^2 G(s)} \right\} \]

Define static acceleration error constant

\[ K_a = \lim_{s \to 0} \left\{ s^2 G(s) \right\} \]

\[ \rightarrow e_{ss} = \frac{1}{K_a} \]
For a system types 0 & 1 $K_a = 0 \rightarrow e_{ss}$ is infinite
Type 2 system $K_a$ is finite $\rightarrow e_{ss}$ is finite
Type 3 or higher system $K_v$ is infinite $\rightarrow e_{ss}$ is zero
Figure 7.5
Feedback control system for Example 7.2
Figure 7.6
Feedback control system for Example 7.3

\[
\begin{align*}
R(s) & \quad + \quad E(s) \\
& \quad - \quad \frac{100(s + 2)(s + 6)}{s(s + 3)(s + 4)} \\
& \quad \quad \quad \quad \quad C(s)
\end{align*}
\]
Figure 7.8
Feedback control system for defining system type

\[
\frac{K(s + z_1)(s + z_2) \cdots}{s^n(s + p_1)(s + p_2) \cdots}
\]
Figure 7.10
Feedback control system for Example 7.6

\[
\begin{align*}
R(s) & \quad + \\
E(s) & \quad \times \\
\frac{K(s + 5)}{s(s + 6)(s + 7)(s + 8)} & \quad C(s)
\end{align*}
\]
# Table 7.2
Relationships between input, system type, static error constants, and steady-state errors

<table>
<thead>
<tr>
<th>Input</th>
<th>Steady-state error formula</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step, $u(t)$</td>
<td>$\frac{1}{1 + K_p}$</td>
<td>$K_p = \frac{1}{1 + K_p}$</td>
<td>$K_p = \infty$</td>
<td>$K_p = \infty$</td>
</tr>
<tr>
<td>Ramp, $tu(t)$</td>
<td>$\frac{1}{K_v}$</td>
<td>$K_v = 0$</td>
<td>$K_v = \frac{1}{K_v}$</td>
<td>$K_v = \infty$</td>
</tr>
<tr>
<td>Parabola, $\frac{1}{2}t^2u(t)$</td>
<td>$\frac{1}{K_a}$</td>
<td>$K_a = 0$</td>
<td>$K_a = 0$</td>
<td>$K_a = \frac{1}{K_a}$</td>
</tr>
</tbody>
</table>

*Control Systems Engineering, Fourth Edition* by Norman S. Nise
Copyright © 2004 by John Wiley & Sons. All rights reserved.
# Steady State Error - A Summary

<table>
<thead>
<tr>
<th>SYSTEM TYPE</th>
<th>Step Input</th>
<th>Ramp Input</th>
<th>Parabolic Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>$\frac{1}{1 + K_P}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$\frac{1}{K_v}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{K_a}$</td>
</tr>
</tbody>
</table>
Figure 7.11
Feedback control system showing disturbance
Figure 7.13
Feedback control system for Example 7.7
Figure 7.14
System for skill-Assessment
Exercise 7.4

\[ R(s) \rightarrow - \rightarrow E(s) \rightarrow + \rightarrow 1000 \rightarrow + \rightarrow s + 2 \rightarrow s + 4 \rightarrow C(s) \]
Figure 7.15
Forming an equivalent unity feedback system from a general nonunity feedback system
Figure 7.16
Nonunity feedback control system for Example 7.8
Figure 7.17
Nonunity feedback control system with disturbance
Figure 7.18
Nonunity feedback system for Skill-Assessment Exercise 7.5
Example 1

- Taken from Ogata, p287
- Consider liquid level control system. Determine the steady state effect of disturbance of size $D_0$ if proportional control is used and alternatively if integral control is used.
Example 1

- **Block Diagram**

![Block Diagram](image)

- Controller
- Disturbance

Symbols:
- $X(s)$
- $E(s)$
- $G_c(s)$
- $G(s)$
- $H(s)$
- $D(s)$
- $G(s)$

Equations:
- $G(s)$
- $X(s)$
- $E(s)$
- $D(s)$

Diagram shows the interaction between the system, controller, and disturbance.
Example 1

\[ G(s) = \frac{R}{RCS + 1}, \]
\[ G_c(s) = \begin{cases} 
K_p & \text{Proportional control} \\
K/s & \text{Integral control}
\end{cases} \]

Disturbance \( D(s) = D_0/s \)
Example 1

Taking variation in setpoint as zero $X(s) = 0$

$$
H(s) = \frac{K_p R}{RCs + 1} E(s) + \frac{R}{RCs + 1} D(s)
$$

$$
E(s) = -H(s) = -\frac{K_p R}{RCs + 1} E(s) - \frac{R}{RCs + 1} D(s)
$$
Example 1

\[ \rightarrow E(s) = - \frac{R}{RCs + 1 + K_pR} D(s) \]

\[ D(s) = \frac{D_0}{s} \]

\[ \rightarrow E(s) = - \frac{R}{RCs + 1 + K_pR} \cdot \frac{D_0}{s} \]
Example 1

$$E(s) = \frac{RD_0}{1 + K_pR} \left( \frac{1}{s + \frac{1+K_pR}{RC}} \right) - \frac{RD_0}{1 + K_pR} \cdot \frac{1}{s}$$

$$e(\infty) = \lim_{s \to 0} \{sE(s)\} = -\frac{RD_0}{1 + K_pR}$$

To get good steady state error need high value of $K_p$
Example 1

if controller is integral $G_c(s) = \frac{K}{s}$

$E(s) = -\frac{Rs}{RCs^2 + s + KR}D(s)$

$\rightarrow e(\infty) = \lim_{s \to 0} \{sE(s)\} = \lim_{s \to 0} \left\{ -s \frac{Rs}{RCs^2 + s + KR} \frac{D_0}{s} \right\} = 0$

Integrator eliminates steady state error due to step disturbance
Homeworks

Ogata Chapter
Nise chapter 7: 4, 12, 16, 35, 38, 45