Kuliah ke – 12
Program S1-Reguler DTE FTUI
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Introduction to Wavelets

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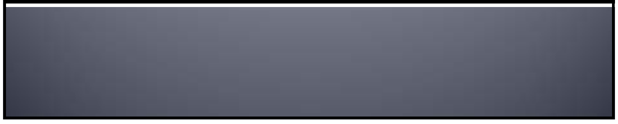
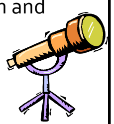


List of topics

- Development in History
- Why transform?
- Why wavelets?
- Wavelets like basis components
- Wavelets examples
- Fast wavelet transform
- Wavelets like filter
- Wavelets advantages

DEVELOPMENT IN HISTORY

- Pre-1930
  - Joseph Fourier (1807) with his theories of frequency analysis
- The 1930s
  - Using scale-varying basis functions; computing the energy of a function
- 1960-1980
  - Guido Weiss and Ronald R. Coifman; Grossman and Morlet
- Post-1980
  - Stephane Mallat; Y. Meyer; Ingrid Daubechies; wavelet applications today



PRE-1930

- Fourier Synthesis
  - Main branch leading to wavelets
  - By Joseph Fourier (born in France, 1768-1830) with frequency analysis theories (1807)
- From the Notion of Frequency Analysis to Scale Analysis
  - Analyzing *f*(*x*) by creating mathematical structures that vary in scale
    - Construct a function, shift it by some amount, change its scale, apply that structure in approximating a signal
    - Repeat the procedure. Take that basic structure, shift it, and scale it again. Apply it to the same signal to get a new approximation
- Haar Wavelet
  - The first mention of wavelets appeared in an appendix to the thesis of A. Haar (1909)
  - With *compact support,* vanishes outside of a finite interval
  - Not continuously differentiable

For any 2π periodical function $f(x)$:

$$f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)dx$$

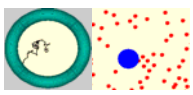
$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)dx$$

Jean-Baptiste-Joseph Fourier (1768-1830)

THE 1930s

- Finding by the 1930s Physicist Paul Levy
  - Haar basis function is superior to the Fourier basis functions for studying small complicated details in the Brownian motion
- Energy of a Function by Littlewood, Paley, and Stein
  - Different results were produced if the energy was concentrated around a few points or distributed over a larger interval

$$Energy = \frac{1}{2}\int_0^{2\pi} |f(x)|^2 dx$$



1960-1980

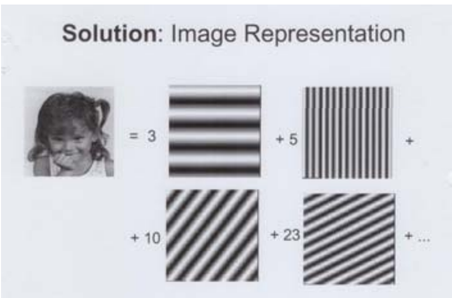
- Created a Simplest Elements of a Function Space, Called Atoms
  - By the mathematicians Guido Weiss and Ronald R. Coifman
  - With the goal of finding the atoms for a common function
- Using Wavelets for Numerical Image Processing
  - David Marr developed an effective algorithm using a function varying in scale in the early 1980s
- Defined Wavelets in the Context of Quantum Physics
  - By Grossman and Morlet in 1980

POST-1980

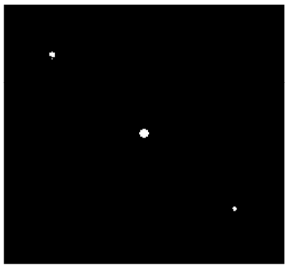
- An Additional Jump-start By Mallat
  - In 1985, Stephane Mallat discovered some relationships between quadrature mirror filters, pyramid algorithms, and orthonormal wavelet bases
- Y. Meyer's First Non-trivial Wavelets
  - Be continuously differentiable
  - Do not have compact support
- Ingrid Daubechies' Orthonormal Basis Functions
  - Based on Mallat's work
  - Perhaps the most elegant, and the cornerstone of wavelet applications today

Why transform?

Image representation

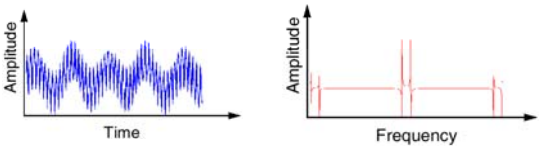


Noise in Fourier spectrum



Fourier Analysis

- Breaks down a signal into constituent sinusoids of different frequencies



In other words: Transform the view of the signal from time-base to frequency-base.

What's wrong with Fourier?

- By using Fourier Transform , we loose the time information : WHEN did a particular event take place ?
- FT can not locate drift, trends, abrupt changes, beginning and ends of events, etc.
- Calculating use complex numbers

Time and Space definition

- Time – for one dimension waves we start point shifting from source to end in time scale
- Space – for image point shifting is two dimensional
- Here they are synonyms

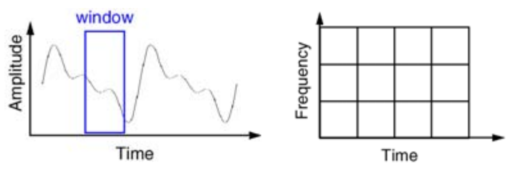
Kronneker function

$$\psi_k(t) = \delta_k(t) = \begin{cases} 1, k = t \\ 0, k \neq t \end{cases}$$

Can exactly show the time of appearance but have not information about frequency and shape of signal.

Short Time Fourier Analysis

- In order to analyze small section of a signal, Denis Gabor (1946), developed a technique, based on the FT and using *windowing* : STFT



STFT (or: Gabor Transform)

- A compromise between time-based and frequency-based views of a signal.
- both time and frequency are represented in limited precision.
- The precision is determined by the size of the window.
- Once you choose a particular size for the time window - it will be the same for all frequencies.

What's wrong with Gabor?

- Many signals require a more flexible approach - so we can vary the window size to determine more accurately either time or frequency

What is Wavelet Analysis ?

- And…what is a wavelet…?

- A wavelet is a waveform of effectively limited duration that has an average value of zero

Wavelet's properties

- Short time localized waves with zero integral value.
- Possibility of time shifting.
- Flexibility.

The Continuous Wavelet Transform (CWT)

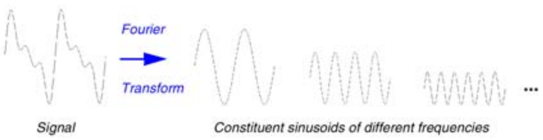
- A mathematical representation of the Fourier transform:

$$F(w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}\,dt$$

- Meaning: the sum over all time of the signal *f(t)* multiplied by a complex exponential, and the result is the Fourier coefficients F(◆)

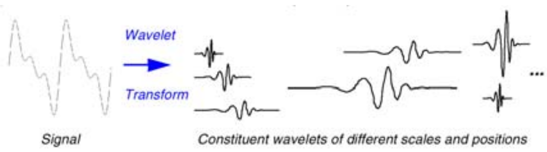
Wavelet Transform (Cont'd)

- Those coefficients, when multiplied by a sinusoid of appropriate frequency ◆ yield the constituent sinusoidal component of the original signal:



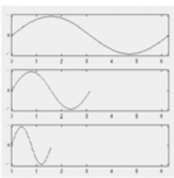
Wavelet Transform

- And the result of the CWT are Wavelet coefficients
- Multiplying each coefficient by the appropriately scaled and shifted wavelet yields the constituent wavelet of the original signal:



Scaling

- Wavelet analysis produces a time-scale view of the signal.
- *Scaling* means stretching or compressing of the signal.
- scale factor (a) for sine waves:



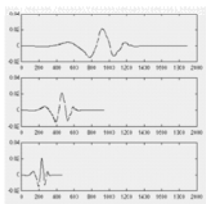
$$f(t) = \sin(t) \;;\; a = 1$$

$$f(t) = \sin(2t) \;;\; a = \tfrac{1}{2}$$

$$f(t) = \sin(4t) \;;\; a = \tfrac{1}{4}$$

Scaling (Cont'd)

- Scale factor works exactly the same with wavelets:



$$f(t) = \Psi(t) \;;\; a = 1$$

$$f(t) = \Psi(2t) \;;\; a = \tfrac{1}{2}$$

$$f(t) = \Psi(4t) \;;\; a = \tfrac{1}{4}$$

Slide 20

s1 student, 11/3/2003

Wavelet function

$$\Psi_{a,b_x,b_y}(x,y)=\frac{1}{|a|}\Psi\left(\frac{x-b_x}{a},\frac{y-b_y}{a}\right)$$

- b – shift coefficient
- a – scale coefficient

$$\Psi_{a,b}(x)=\frac{1}{\sqrt{a}}\Psi\left(\frac{x-b}{a}\right)$$

- 2D function

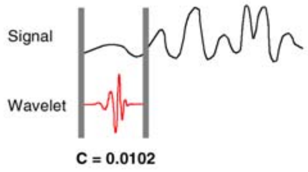
CWT

- Reminder: The *CWT* is the sum over all time of the signal, multiplied by scaled and shifted versions of the wavelet function

Step 1:
Take a Wavelet and compare it to a section at the start of the original signal

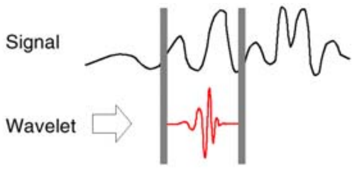
CWT

Step 2:
Calculate a number, C, that represents how closely correlated the wavelet is with this section of the signal. The higher C is, the more the similarity.



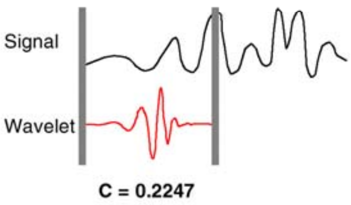
CWT

- Step 3: Shift the wavelet to the right and repeat steps 1-2 until you've covered the whole signal



CWT

- Step 4: Scale (stretch) the wavelet and repeat steps 1-3



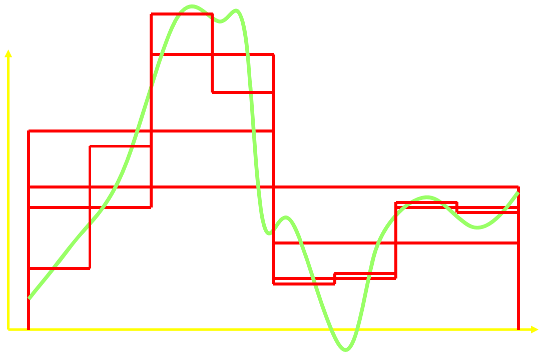
Wavelets examples
Dyadic transform

- For easier calculation we can to discrete continuous signal
- We have a grid of discrete values that called dyadic grid
- Important that wavelet functions compact (eg. *no overcalculatings*)

$$a=2^j$$

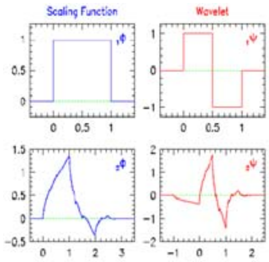
$$b=k2^j$$

Haar transform



Wavelet functions examples

- Haar function
- Daubechies function



Properties of Daubechies wavelets

I. Daubechies, *Comm. Pure Appl. Math.* 41 (1988) 909

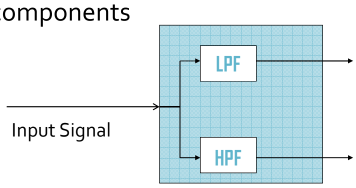
- Compact support
  - finite number of filter parameters/fast implementations
  - high compressibility
  - fine scale amplitudes are very small in regions where the function is smooth/sensitive recognition of structures
- Identical forward/backward filter parameters
  - fast, exact reconstruction
  - very asymmetric

Mallat* Filter Scheme

- Mallat was the first to implement this scheme, using a well known filter design called "two channel sub band coder", yielding a *'Fast Wavelet Transform'*

Approximations and Details:

- Approximations: High-scale, low-frequency components of the signal
- Details: low-scale, high-frequency components

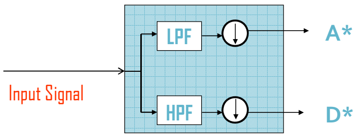


Decimation

- The former process produces twice the data it began with: N input samples produce N approximations coefficients and N detail coefficients
- To correct this, we *Down sample* (or: *Decimate)* the filter output by two, by simply throwing away every second coefficient

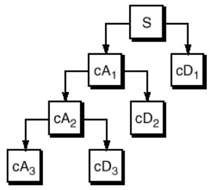
Decimation (cont'd)

So, a complete one stage block looks like:



Multi-level Decomposition

- Iterating the decomposition process, breaks the input signal into many lower-resolution components: *Wavelet decomposition tree*:



Orthogonality

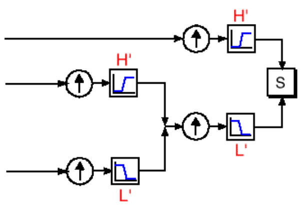
- For 2 vectors $\langle v, w \rangle = \sum_n v_n w_n^* = 0$
- For 2 functions $\langle f(t), g(t) \rangle = \int_a^b f(t) g^*(t) dt = 0$

Why wavelets have orthogonal base?

- Easier calculation
- When we decompose some image and calculating zero level decomposition we have accurate values
- Scalar multiplication with other base function equals zero

Wavelet reconstruction

- Reconstruction (or synthesis) is the process in which we assemble all components back



Up sampling (or interpolation) is done by zero inserting between every two coefficients

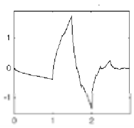
Wavelets like filters

Relationship of Filters to Wavelet Shape

- Choosing the correct filter is most important
- The choice of the filter determines the shape of the wavelet we use to perform the analysis

Example

- A low-pass reconstruction filter (L') for the db2 wavelet:



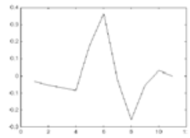
The filter coefficients (obtained by Matlab *dbaux* command):
`0.3415 0.5915 0.1585 -0.0915`
reversing the order of this vector and multiply every second coefficient by -1 we get the high-pass filter H':
`-0.0915 -0.1585 0.5915 -0.3415`

Example (Cont'd)

- Now we up-sample the H' coefficient vector:

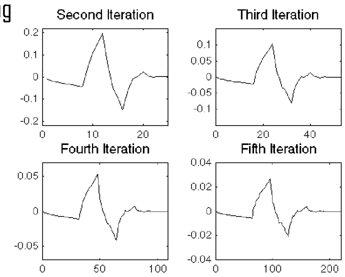
`-0.0915 0 -0.1585 0 0.5915 0 -0.3415 0`

- and Convolving the up-sampled vector with the original low-pass filter we get:



Example (Cont'd)

Now iterate this process several more times, repeatedly up-sampling and convolving the resultant vector with the original low-pass filter, a *pattern* begins to emerge:



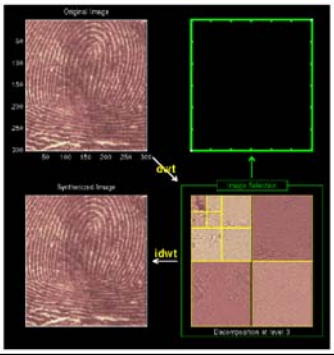
Example: Conclusion

- The curve begins to look more like the *db2* wavelet: the wavelet shape is determined entirely by the coefficient of the reconstruction filter
- You can't choose an arbitrary wavelet waveform if you want to be able to reconstruct the original signal accurately!

Compression Example

- A two dimensional (image) compression, using 2D wavelets analysis
- The image is a Fingerprint
- FBI uses a wavelet technique to compress its fingerprints database

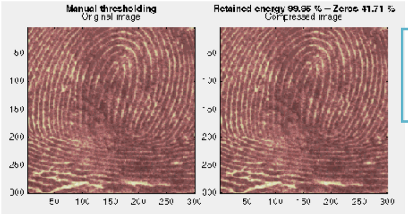
Fingerprint compression



Wavelet: Haar
Level: 3

Result

Original Image Compressed Image



Threshold: 3.5
Zeros: 42%
Retained energy: 99.95%