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Analysis Of Projection Optimization In Compressive Sensing Framework Into Reconstruction Performance

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Abstract— **Compressive Sensing (CS), which is firmly mathematically formulated by Danoho D, Candes E, Romberg J, and Tao T, is much developed especially for sensing and signal reconstruction. Its advantage framework on reducing number of measurement data while maintaining the performance of reconstruction quality, makes many researchers concern on developing the compressive sensing performance. The main parameters in CS are projection matrix and sparse base representation (dictionary). Subject to Restricted Isometric Property, the more incoherence between projection matrix and the dictionary, the more precise the signal reconstruction. Thus, a number of fundamental researches regarding projection optimization to optimize the incoherence between projection matrix and the dictionary have been developed. This paper elaborate the analysis of projection optimization's impact into reconstruction performance on signal with random and structured projection matrix. The simulations show that the projection optimization does not always imply better reconstruction especially for signal reconstruction with structured projection matrix.**

Keywords—Compressive Sensing, Projection Optimization

I. Introduction

Compressive Sensing is much developed since its ability to reduce the sampling measurement while maintaining the quality of the reconstruction. Measurement data always be the central issue for almost recovery or reconstruction problem. Limited hardware is normally makes limitation on capturing the measurement data of the environment in the sensing domain. CS framework enable to minimize it.

Sparsity and incoherence are two fundamental principle on CS framework [1]. In CS framework, signal can be precisely recovered if the sensing signal is naturally sparse or sparse in a transformed domain by certain transformed function or dictionary [1-4]. The second principle need to be concerned is the incoherency between the projection (measurement) function and the sparse dictionary [1]. Based on mathematical principle on RIP (Restricted Isotropy Property), the more incoherence, the more accurate reconstruction will be achieved [2]. Subject to the RIP theory, some fundamental researches on doing CS optimization have been published [2, 3, 5-8]. Main objective of those proposed CS optimization algorithm is to minimize the incoherence between the projection matrix and the dictionary. It can be proceed by either optimizing the projection matrix or the dictionary design. The publications regarding this issues show that updating projection matrix to minimize the incoherence value succeeds to present better reconstruction accuracy. However most of them are tested on simple signal which is commonly apply random projection matrix. In another hand, real signal recovery or reconstruction is not always simple. They mostly employ non-random (structured) projection matrix. In addition they are high under sampled.

This paper will evaluate the impact of projection matrix optimization in CS performance not only in simple signal reconstruction but also in more complicated reconstruction case. The simulations show that projection matrix optimization will always reduce the incoherence but does not always correlate to the better reconstruction accuracy especially for reconstruction system with structured projection matrix.

II. Methodology

A. Compressive Sensing

CS states that any signal that is naturally sparse or sparse in certain domain can be exactly recovered from number of signal's sampling which its dimension is considerably lower than the number of sampling required in Shanon-Nyquist theorem [1, 2, 4]. Mathematically, this concept is another approach to solve a linear system which has less number of equations compared to the number of variables should be solved which is termed as undetermined linear system.

The Mathematical Model

Given a discrete-time signal $x \epsilon R^N$ and consider a measurement system that acquire M- dimension of measurement value, then mathematically the linear measurement can be represented as [2]:

$$y = \Phi x \quad (1)$$

where $\Phi \epsilon R^{MxN}$ and $y \epsilon R^M$. Φ represents the measurement or sensing matrix. M is typically is much smaller compared to N. $x \epsilon R^N$ is a coefficient vector which normally has only $K \ll N$ non-zero coefficient [2]. In order to ensure that the original signal is properly adapted with the compressive sensing, the original signal x is often reformulated as a linear combination of a small number of signals taken from a "resource database"

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determined as dictionary $\psi \epsilon R^{NxL}$ [2]. Element of dictionary is typically unit norm function called atom [2].

$$x = \psi s \quad (2)$$

Then x is determined as sparse signal in base ψ with K-degree of sparse.

By representing the original signal into certain dictionary, the linear measurement on Eq. 1 can be represented as:

$$y = \Phi \, \psi s \quad (3)$$

The main idea of CS system is projection of x to a low dimensional measurement vector y by measurement matrix Φ which is completely has no relation to the sparse base ψ [1, 2, 4]. However, some of mathematical works and former simulations have shown criteria that should be satisfied by the measurement matrix in correlation with the sparse base matrix to achieve good CS performance.

The mathematical model given in Eq.3 indicates that the

Mutual Coherence

Mutual coherence of $A = \Phi \, \psi$, denoted as $\mu(A)$, determine the worst case coherence between any two column (atoms) of A

Definition 1

For a given matrix $A = \Phi \, \psi$, the mutual coherence of A $\{\mu(A)\}$ is defined as the largest absolute and normalized inner product between the two different column in A formulated as measurement matrix and the sparse base (dictionary) matrix are the crucial parameters in CS system. Thus, most of proposing algorithm on improving CS system performance is by defining optimization procedure for both of those parameters.

The Fundamental Principle of Compressive Sensing

Compressive sensing principles concerning on some aspects that effect into the compressive sensing's performance. By acknowledging those properties, the optimal performance of compressive sensing should be achieved. Two fundamental principles that should be pointed for CS performance are the sparsity and the incoherence principle [2, 4].

Sparsity

In compressive sensing framework, the reconstruction of a signal can be exact if the signal being sensed has a low information rate [2]. In another words, it is sparse in the original or other transformed domain. Thus, sparsity is one of the crucial properties in compressive sensing framework. Normally, to make sure that sparsity of the signal can be reliably adaptable with the compressive sensing framework, the sensed signal is transformed into certain domain by certain sparse base function that is called as dictionary [2]. Some dictionaries commonly used are cosine base, sine base, wavelet base, chirplet base, curvelet base, etc [9].

The Incoherence

The second most important principle of CS framework on achieving optimal performance is incoherence principle. The more incoherence between the measurement matrix and the dictionary, the more optimal reconstruction should be achieved. Mathematically, the incoherence principle is represented by Restricted Isotropy Property [2].

Restricted Isotropy Property

Restricted Isotropy Property is a property that should be satisfied by the measurement (projection) matrix Φ to guarantee the convergence of the reconstruction algorithm that recover any K-sparse signal by using M measurement value. Mathematically, for any K-sparse signal s and any constant $\varepsilon \in (0,1)$, the RIP criterion restrict the Φ, ψ into following criteria

$$1 - \varepsilon \leq \frac{\|\Phi \, \psi s\|_2}{\|s\|_2} \leq 1 + \varepsilon \quad (4)$$

Interpretation of the above criterion is the **un-correlation** (incoherency) between the measurement (projection) matrix and the sparse base matrix (dictionary).

The other way to see the un-correlation between the measurement (projection) matrix and the sparse base matrix is by observing the mutual coherence between those two parameters by using Gram Matrix [2].

$$\mu(A) \cong \max_{1 \leq i \neq j \leq L} \frac{|A_i^T A_j|}{\|A_i\|_2 \|A_j\|_2} \quad (5)$$

For a given matrix $A = \Phi \, \psi$, the Gram matrix is defined as:

$$G = A^T A \quad (6)$$

the (i,j)-th element of Gram matrix of A is defined as:

$$g_{ij} = A_i^T A_j \quad (7)$$

The Gram matrix is normalized such that $g_{ij} = 1, for \; \forall i = j$. The mutual coherence of A is determined by the maximum value of the off-diagonal element of G.

The value of the mutual coherence are bounded in the interval of $\underline{\mu} \leq \mu(A) \leq 1$, with low bound $\underline{\mu}$ is defined as [2]

$$\underline{\mu} \cong \sqrt{\frac{L - M}{M(L - 1)}} \quad (8)$$

Instead of the maximum value of the off-diagonal element of G, the former simulations show that the average of the mutual coherence is more related to the performance of a CS system. Thus, the other measurement, called average mutual coherence, is drawn as follow [2]:

$$\bar{\mu}(A) = \frac{\sum_{\forall(i,j),with\ i\neq j} g_{ij}}{N_t} \quad (9)$$

With N_t number of the off-diagonal element. On this paper, the distribution of the off-diagonal entry of normalized gram matrix G is presented to give supporting reasons of the resulted simulation performance.

B. Optimization on CS Performance

Subject to the RIP condition, the performance of Compressive Sensing on reconstruction problem can be indicated by its incoherence value between the sparse base (dictionary) and the projection matrix. As theoretically revealed, the more incoherence between the sparse base and projection matrix, the more accurate the reconstruction. Thus, the possibility to improve the CS performance is much related to the either optimization of the dictionary or the projection matrix design (determination) as conceived on schematic diagram, in Figure 1

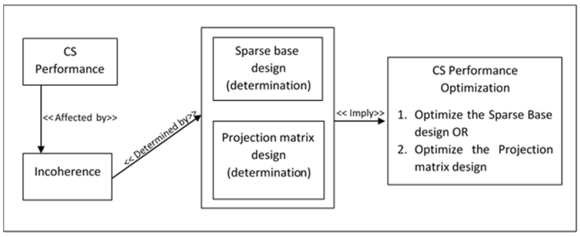


Figure 1. CS Optimization based on RIP

On the presented research, the performance analysis will be subject to optimization on the projection matrix.

C. Projection Optimization based on ETF

CS performance is mainly determined by the low mutual coherence between the projection matrix and the dictionary. The mutual coherence is used as the performance parameter since it is proportional with the number of data needed to recover the signal. Most of the CS framework assume the projection matrix to be random. Projection optimization proposed by Xu et al is based on Equiangular Tight Frame (ETF)[2]. It possibly the projection matrix to be updated as close as to ETF design which has minimum coherence [2]. The algorithm of ETF is presented on Algorithm 1 below [2]:

Algorithm 1. Projection Optimization based on ETF (Xu et al)

Initialization

- *Initial Projection matrix Φ*
- *Sparse base (dictionary) ψ*
- *Calculate the equivalent dictionary $D = \Phi * \psi$*

Iteration Procedure

Loop: Set k=1 and repeat iter times

While $1 \leq k \leq iter$

1. *Define Gram matrix G,* $G = D' * D$, *define every element of Gram matrix G as* g_{ij}
2. *Project the Gram matrix into a convex set Λ^k, with below criteria*

$$g_{ij} = \begin{cases} 1, & i \neq j \\ g_{ij,} & abs(g_{ij}) < \mu_g \\ sign(g_{ij,}).\mu_g & else \end{cases} \quad (10)$$

3. *Update the Gram matrix*

$$G_{k+1} = \alpha G_k + (1-\alpha)G_{k-1}, \quad 0 < \alpha < 1 \quad (11)$$

4. *Update the projection matrix $\Phi$and by Single Value Decomposition (SVD) the optimized projection matrix Φ_{new} is obtained*

END

The optimized projection matrix resulted is reported to be able to reduce the necessary number of samples for reconstruction improvement over the reconstruction accuracy [2]. It is also reported that this optimization algorithm is also bring benefit for both basis pursuits and orthogonal matching pursuit reconstruction algorithm [2].

D. Simulation Set up

A simulation is set up to evaluate the impact of projection matrix optimization into reconstruction performance in CS framework. Two study cases represent signal reconstruction system by random projection matrix and structured projection matrix. Electrical Capacitance Volume Tomography (ECVT) signal reconstruction is used to represent the sensing system by structured projection matrix. Both of the study cases will utilize synthetic data.

As general the simulation will compare and analyze the performance of CS into signal reconstruction without and with projection matrix optimization on the given study cases. The scenario of the simulation can be explained through block diagram on figure 2 and figure 3 below

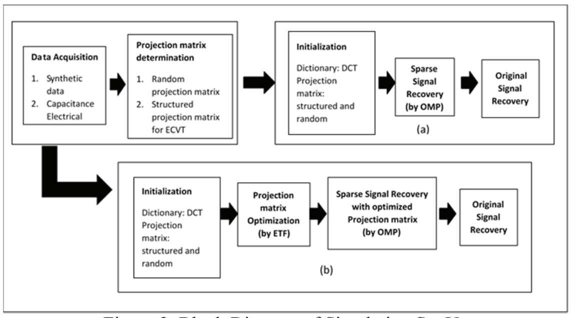


Figure 2. Block Diagram of Simulation Set Up
(a). Non Optimized CS, (b) CS with Projection Optimization

Orthogonal Matching Pursuit (OMP) is used as the reconstruction algorithm to recover or to reconstruct the sparse representation signal for both scenario.

ECVT imaging is used to represent reconstruction system using structured projection matrix. The projection matrix in the imaging system is constructed by certain formula adapted to the ECVT measurement process. Cylindrical and hemisphere sensor are used on the simulation. As detail the simulation set up for ECVT imaging can be described on given block diagram below

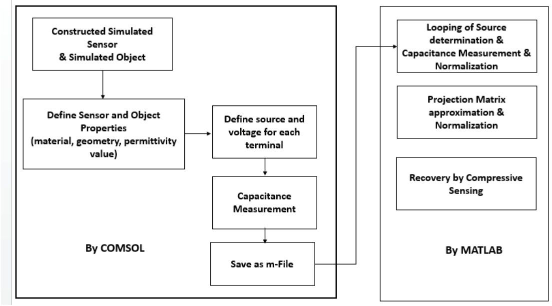


Figure 3. Simulation Set Up for ECVT Imaging

A Glance about ECVT

ECVT is a tomography technique which is based on capacitance measurement [10]. The aim of it is to predict the permittivity distribution inside the sensing domain utilizing the capacitance measurement on the sensor boundary [10]. The technology which is formerly invented by Warsito et al at 2007, is a further invention of ECT that can do volumetric imaging. The technology is able to do 3D object imaging directly that make it more reliable for more rapid observation compared to ECT. The basic difference of ECT and ECVT can be seen clearly when it comes to produce reconstruction for 3D object. 3D reconstruction by ECT projection is generated by stacking up every slice of 2D image. In ECVT, the 3D image reconstruction is generated directly without stacking procedure and it can be real time reconstruction .To produce the 3D reconstruction, ECVT utilizes the fringing effect which is dependent to the sensor design [10].

Naturally, the characteristics of ECT or ECVT signal is sparse. Thus, they CS framework is promising to solve image reconstruction on ECT and ECVT. However, to make sure the sparsity meet the CS framework requirement, spare base is required to represent the sparse signal.

The mathematical model

Linearization on ECVT measurement system can be mathematically modeled as [10]:

$$C = SG \tag{12}$$

where C is *M*-dimension of measured capacitance vector, *G* is N-dimension of permittivity distribution vector and *S* is *M* x *N* dimension of sensitivity matrix. The sensitivity matrix is the projection matrix which is determined by [10]

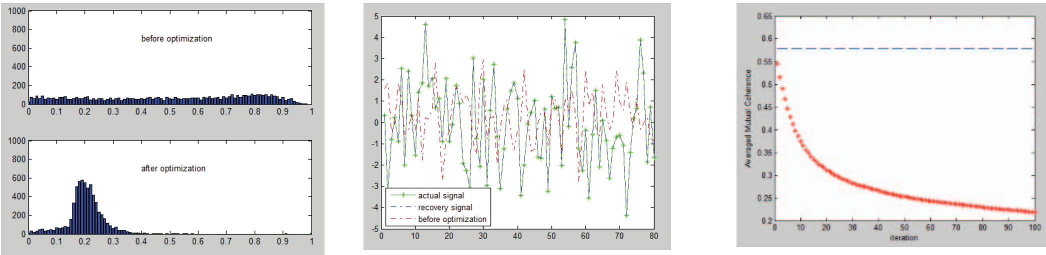
$$S_{ij} \cong V_{0j} \frac{E_{si}(x,y,z).E_{di}(x,y,z)}{V_{si}V_{di}} \tag{13}$$

where $E_{si}(= -\nabla\phi)$ is the electrical field distribution vector when the source electrode in the *i*th pair is activated with voltage V_{si} while the rest of the electrodes are grounded. E_{di} is the electrical field distribution vector when the detector electrode in the *i*th pair is activated with voltage V_{di} while the rest of the electrodes are grounded. V_{0j} is the volume of the *j*th voxel. The resulted projection matrix is not random and considered to be structured.

III. Result and Discussion

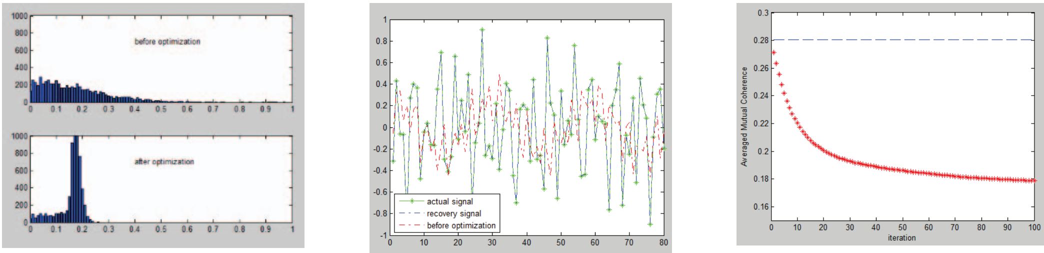
The CS with optimization is simulated into some study cases to observe the optimization impact into mutual coherence behavior which imply into the reconstruction performance. First study case is signal reconstruction for arbitrary synthetic signal with random projection matrix as presented on Figure 4 and Figure 5. Figure 4 and Figure 5 present the performance of reconstruction with random dictionary and DCT correspondingly. Length of the signal is set up to be 80 with 28 measurement data and cardinality 4.

Figure 4 (a) and Figure 5(a) show that optimization on projection matrix succeed to decrease the mutual coherence. The distribution of the mutual coherence value is shifted approaching zero. It implies into the reconstruction performance by OMP which is presented on Figure 4(b) and Figure 5(b).



(a). Dist. Off-Diagonal on Gram Matrix (b). Signal Reconstruction (c). Average of Mutual Coherence

Figure 4 Evaluation on Optimized CS on Arbitrary synthetic signal with dictionary Random Normal N=80, K=80, T=4, M=28, $\bar{\mu}_{ETF} = 0.2773 \bar{\mu}_0 = 0.5778$



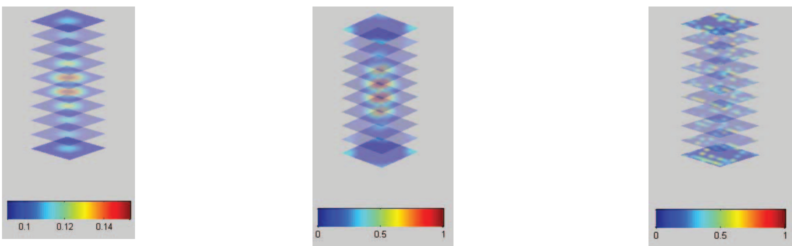
(a). Off-Diagonal Distribution on Gram Matrix (b). Signal Reconstruction (c). Average of Mutual Coherence

Figure 5 Evaluation on Optimized CS on Arbitrary synthetic signal with dictionary DCT
N=80, K=80, T=4, M=28,
$\bar{\mu}_{ETF} = 0.1931, \bar{\mu}_0 = 0.2804$

Figure 4 (a) and Figure 5(a) show that optimization on projection matrix succeed to decrease the mutual coherence. The distribution of the mutual coherence value is shifted approaching zero. It implies into the reconstruction performance by OMP which is presented on Figure 4(b) and Figure 5(b). The reconstruction signal after optimization is much more precise compared to the reconstruction signal before optimization. It can be concluded that the optimization procedure based on ETF to update the projection matrix succeed to improve the reconstruction performance on signal reconstruction with random projection matrix initiation.

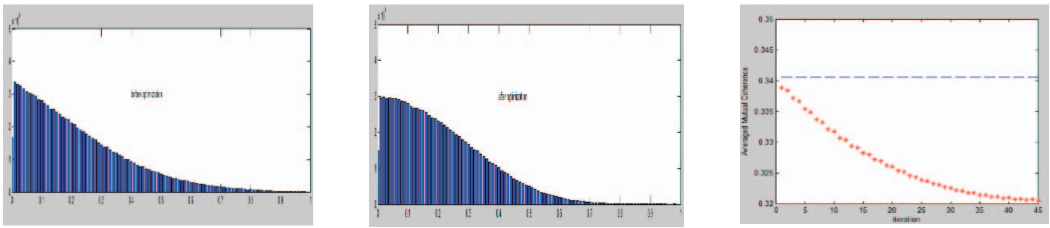
However, in application, signal or image reconstruction is not always working with random projection matrix. Even mostly should be derived from structured projection matrix which adapt to the physical condition of the signal or image reconstruction system.

Figure 6 and Figure 7 present the evaluation of projection matrix optimization on ECVT image reconstruction. As mentioned before, the projection matrix for ECVT is structured instead of random.



(a) (b) (c)

Figure 6. ECVT Reconstruction by 10 x 10 x 10 resolution
(a). ILBP, (b). *Conventional* CS, (c). *Conventional* CS – ETF



Average Mutual Coherence = 0.3407
(a). Before Optimization

Average Mutual Coherence = 0.3265
(b) After Optimization

Average of Mutual Coherence Movement
on ECVT Imaging

Figure 7. Performance by Average Mutual Coherence Parameter

Qualitatively as presented on Figure 6(a) and Figure 6(b), Conventional CS or CS without projection matrix optimization succeed to reduce the elongation error that is still presented by the existing reconstruction method, Iterative Learning Back Projection (ILBP). However artifacts are still exist at the edge of the sensor. Former scientific publications have been justified that reconstruction by CS with fix dictionary on high under-sample measurement system will lead to produce artifact or noise [11]. ECVT measurement system is high under sample, thus what the simulation presented is equitable.

In another hand, Optimization on Projection Matrix lead to fail reconstruction as presented on Figure 6(c). One of the CS performance parameter is incoherence between the projection matrix and dictionary. Figure 7 present the performance measurement by average of off diagonal matrix component on Gram matrix (average of mutual coherence). It is presented that the projection optimization do not significantly reduce the mutual coherence. After 45 iteration it only give 0.02 improvement. It becomes make sense that the reconstruction should not be better after projection matrix optimization. However, fail reconstruction as presented on Figure 6(c) possibly lead to early insight that optimizing CS performance by projection matrix optimization is not working for reconstruction system with structured projection matrix.

IV. Conclusion

The presented paper have analyzed the impact of projection matrix optimization into signal or image reconstruction based on structured and random projection matrix. The conventional CS and CS with the projection matrix optimization by ETF is simulated into signal reconstruction with random projection matrix and signal reconstruction with structured projection matrix. The results is presented on Figure 4 until Figure 7.

Based on the presented figures 4 and figure 5, the projection matrix optimization succeed to decrease the mutual coherence significantly. It happends to both random and DCT dictionary. As the CS theory stated, the smaller mutual coherence lead to better reconstruction as presented on Figure 4(b) and Figure 5(b). In another hand, projection matrix optimization is not practically improve the reconstruction performance of system by non-random or structured projection matrix. Figure 7 show that the projection optimization do not significantly decrease the mutual coherence. Thus, theoritically, it will not improve the reconstruction or reconstruction performance. Applying the projection optimization into ECVT image reconstruction proceed reconstruction failure.

The reconstruction failure as presented on Figure 6(c) possibly lead to early insight that optimizing CS performance by projection matrix optimization is not working for reconstruction system with structured projection matrix. However, it still need further theoretical investigation to support the insight. The insight may imply to dictionary learning is more efficient to improve the CS performance into reconstruction system with structured projection matrix instead of projection matrix optimization.

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