Digital Communication
Signal Space and Average Error Probability

Lecture-4 and 5

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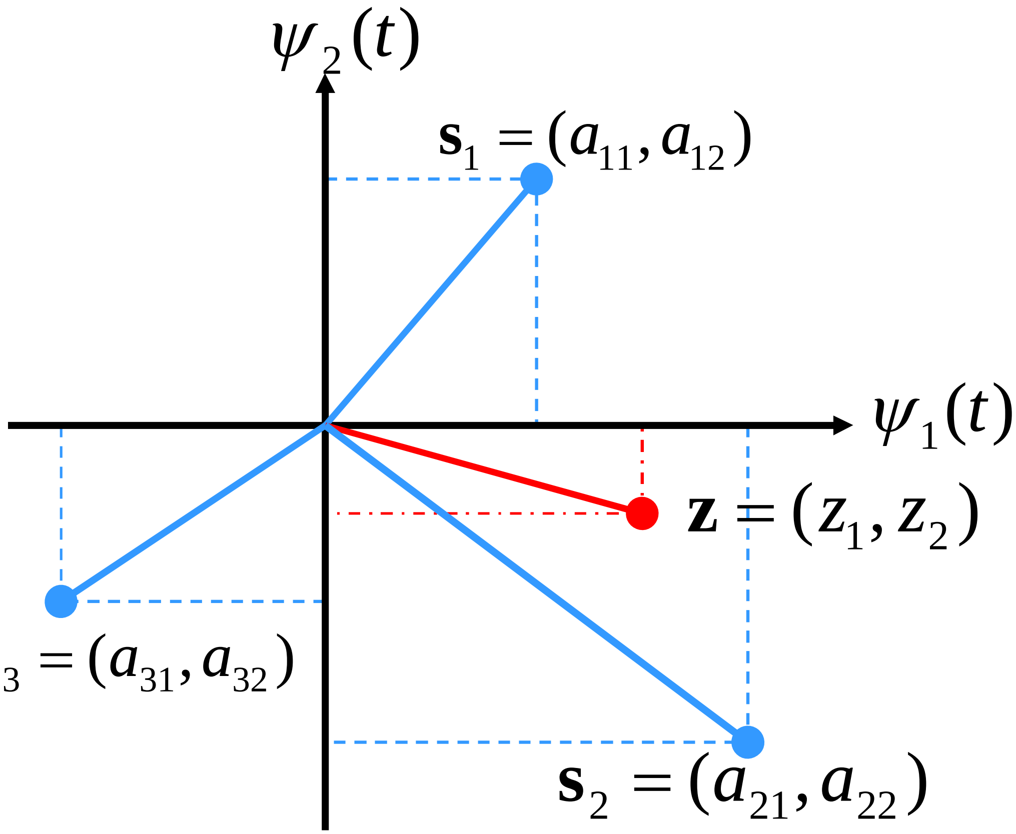
Signal space

- What is a signal space?
  - Vector representations of signals in an N-dimensional orthogonal space
- Why do we need a signal space?
  - It is a means to convert signals to vectors and vice versa.
  - It is a means to calculate signals energy and Euclidean distances between signals.
- Why are we interested in Euclidean distances between signals?
  - For detection purposes: The received signal is transformed to a received vectors. The signal which has the minimum distance to the received signal is estimated as the transmitted signal.

Schematic example of a signal space



Transmitted signal alternatives

$$s_1(t) = a_{11}\psi_1(t) + a_{12}\psi_2(t) \Leftrightarrow \mathbf{s}_1 = (a_{11}, a_{12})$$

$$s_2(t) = a_{21}\psi_1(t) + a_{22}\psi_2(t) \Leftrightarrow \mathbf{s}_2 = (a_{21}, a_{22})$$

$$s_3(t) = a_{31}\psi_1(t) + a_{32}\psi_2(t) \Leftrightarrow \mathbf{s}_3 = (a_{31}, a_{32})$$

Received signal at matched filter output

$$z(t) = z_1\psi_1(t) + z_2\psi_2(t) \Leftrightarrow \mathbf{z} = (z_1, z_2)$$

Signal space

- To form a signal space, first we need to know the inner product between two signals (functions):
  - Inner (scalar) product:

$$< x(t), y(t) >= \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

= cross-correlation between x(t) and y(t)

  - Properties of inner product:

$$< ax(t), y(t) >= a < x(t), y(t) >$$

$$< x(t), ay(t) >= a^* < x(t), y(t) >$$

$$< x(t) + y(t), z(t) >=< x(t), z(t) > + < y(t), z(t) >$$

Signal space …

- The distance in signal space is measure by calculating the norm.
- What is norm?
  - Norm of a signal:

$$\|x(t)\| = \sqrt{< x(t), x(t) >} = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2 dt} = \sqrt{E_x}$$

= "length" of x(t)

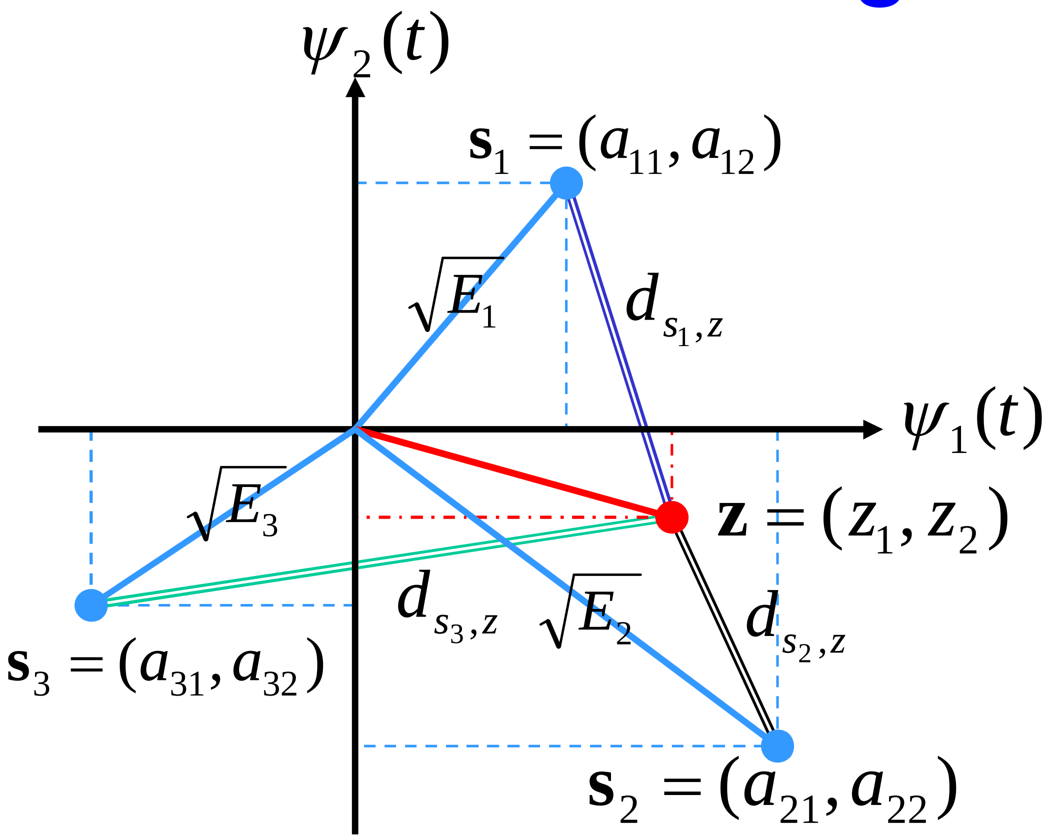
$$\|ax(t)\| = |a| \|x(t)\|$$

  - Norm between two signals:

$$d_{x,y} = \|x(t) - y(t)\|$$

- We refer to the norm between two signals as the Euclidean distance between two signals.

Example of distances in signal space



The Euclidean distance between signals *z(t)* and *s(t)*:

$$d_{s_i,z} = \| s_i(t) - z(t) \| = \sqrt{(a_{i1} - z_1)^2 + (a_{i2} - z_2)^2}$$

$$i = 1,2,3$$

Orthogonal signal space

- N-dimensional orthogonal signal space is characterized by N linearly independent functions $\{\psi_j(t)\}_{j=1}^{N}$ called basis functions. The basis functions must satisfy the orthogonality condition

$$<\psi_i(t), \psi_j(t)> = \int_0^T \psi_i(t)\psi_j^*(t)dt = K_i\delta_{ji} \qquad \begin{matrix} 0 \leq t \leq T \\ j, i = 1, \ldots, N \end{matrix}$$

where

$$\delta_{ij} = \begin{cases} 1 \rightarrow i = j \\ 0 \rightarrow i \neq j \end{cases}$$

- If all $K_i = 1$, the signal space is orthonormal.

Signal space …

- Any arbitrary finite set of waveforms $\{s_i(t)\}_{i=1}^{M}$

where each member of the set is of duration *T*, can be expressed as a linear combination of N orthonogal waveforms $\{\psi_j(t)\}_{j=1}^{N}$ where $N \le M$.

$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t) \qquad i = 1,\ldots,M \qquad N \le M$$

where

$$a_{ij} = \frac{1}{K_j} < s_i(t), \psi_j(t) >= \frac{1}{K_j}\int_0^T s_i(t)\psi_j^*(t)dt \qquad j = 1,\ldots,N \quad i = 1,\ldots,M \qquad 0 \le t \le T$$

$$\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})$$

Vector representation of waveform

$$E_i = \sum_{j=1}^{N} K_j \left|a_{ij}\right|^2$$

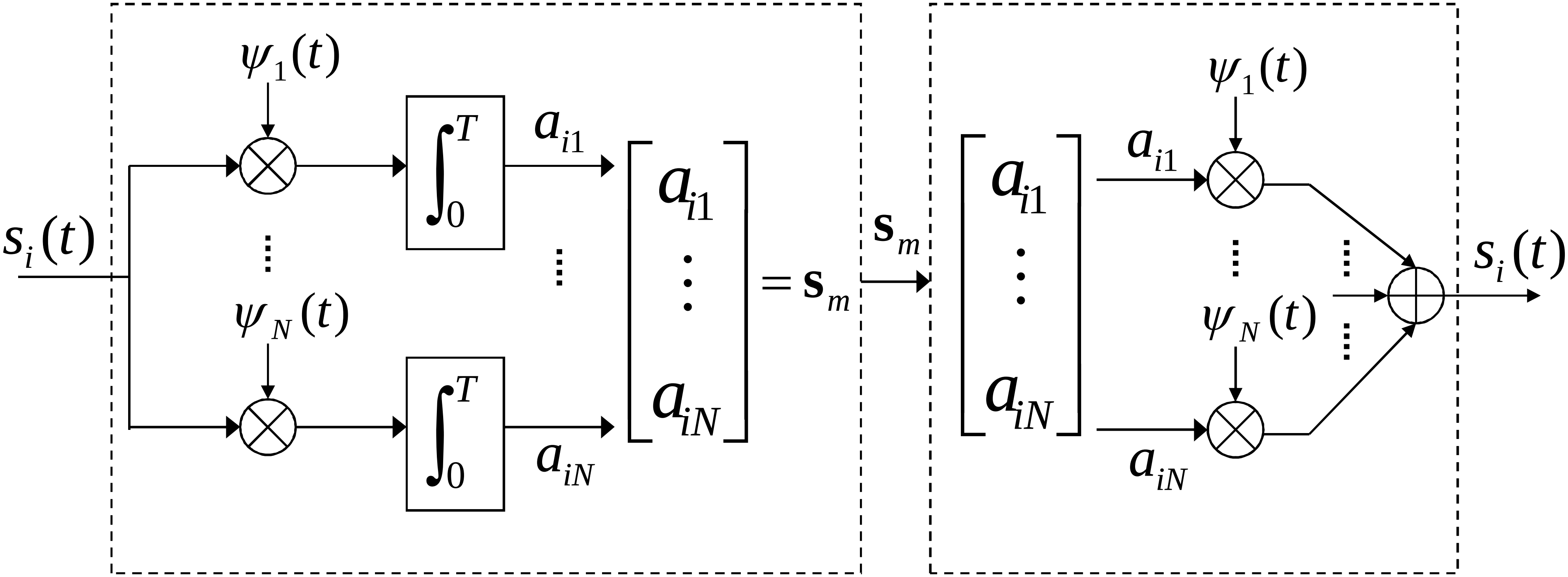
Waveform energy

Signal space …

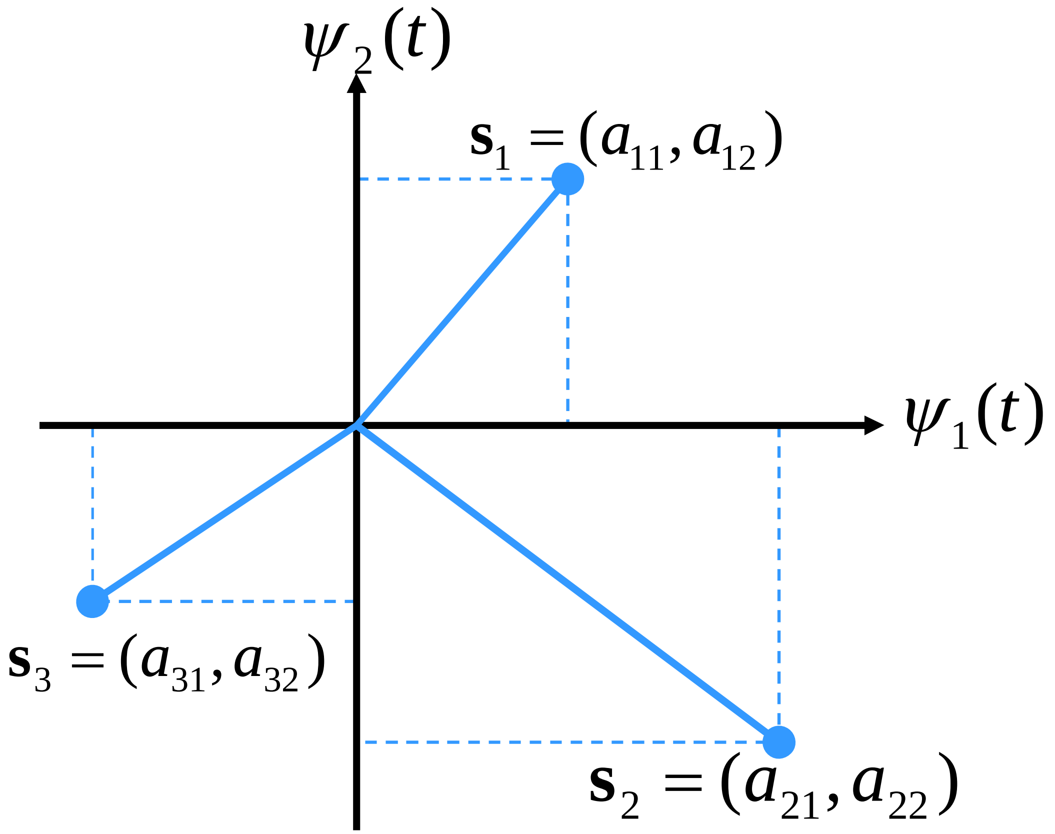
$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t)$$

$$\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})$$

Waveform to vector conversion / **Vector to waveform conversion**



Example of projecting signals to an orthonormal signal space

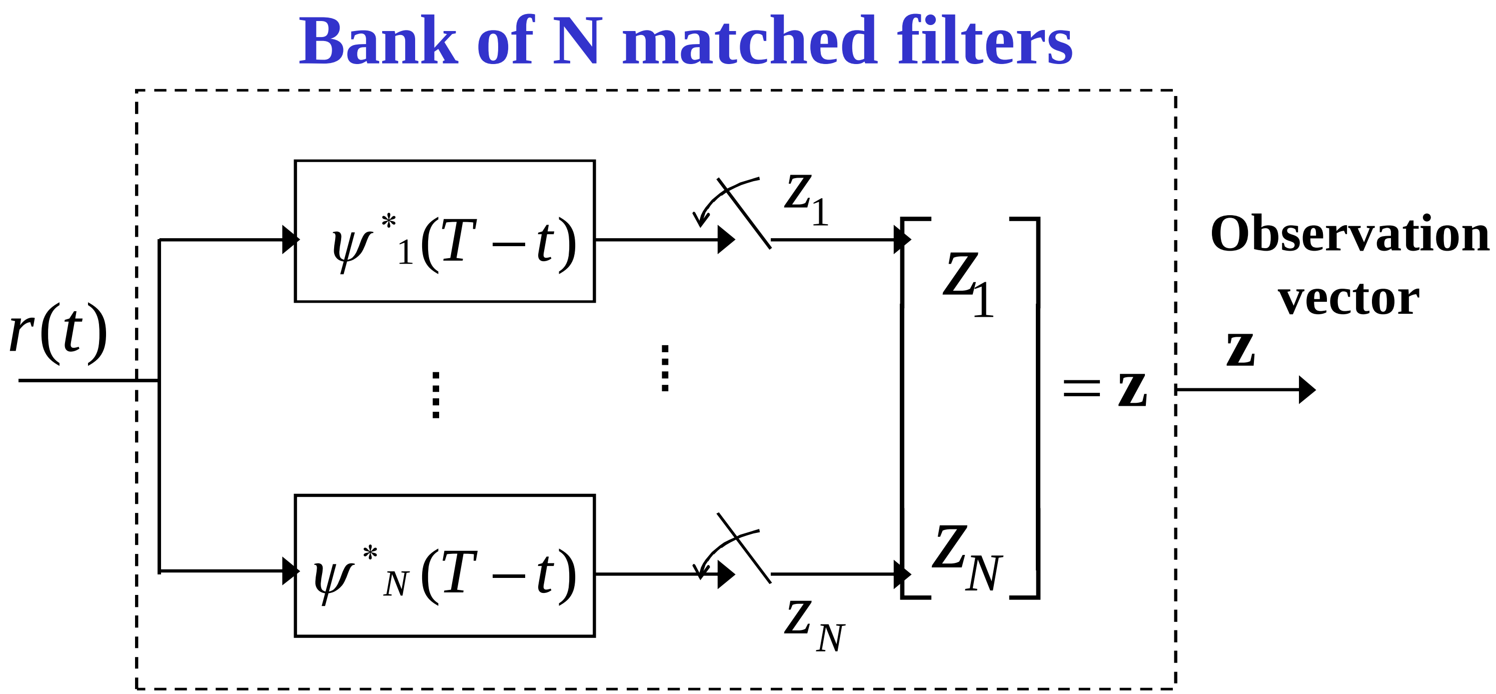


Transmitted signal alternatives

$$\begin{cases} s_1(t) = a_{11}\psi_1(t) + a_{12}\psi_2(t) \Leftrightarrow \mathbf{s}_1 = (a_{11}, a_{12}) \\ s_2(t) = a_{21}\psi_1(t) + a_{22}\psi_2(t) \Leftrightarrow \mathbf{s}_2 = (a_{21}, a_{22}) \\ s_3(t) = a_{31}\psi_1(t) + a_{32}\psi_2(t) \Leftrightarrow \mathbf{s}_3 = (a_{31}, a_{32}) \end{cases}$$

$$a_{ij} = \int_0^T s_i(t)\psi_j(t)dt \qquad j = 1,\ldots,N \qquad i = 1,\ldots,M \qquad 0 \le t \le T$$

Implementation of matched filter receiver



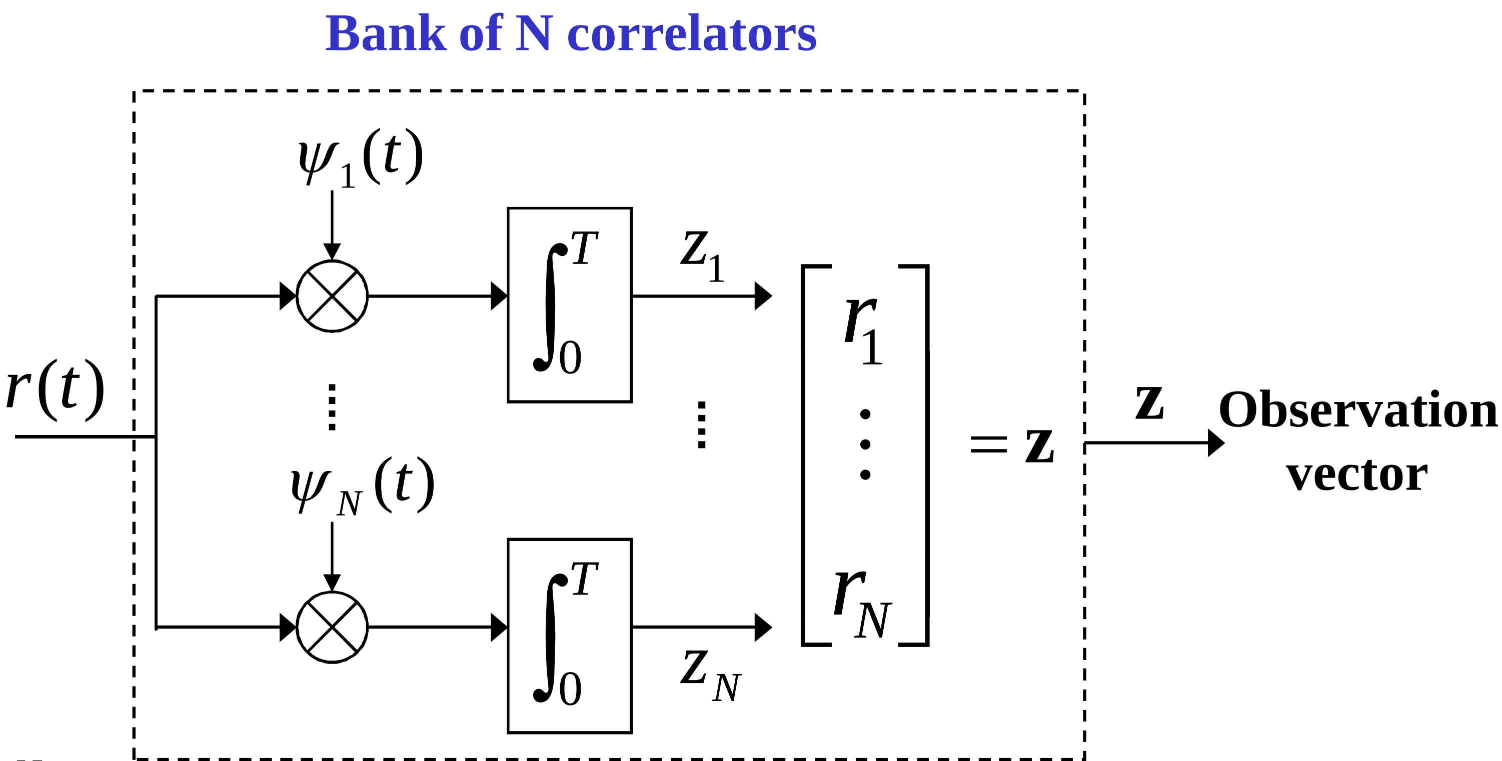
$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t) \qquad i = 1,\ldots,M$$

$$\mathbf{z} = (z_1, z_2, \ldots, z_N)$$

$$N \leq M$$

$$z_j = r(t) * \psi_j(T-t) \qquad j = 1,\ldots,N$$

Implementation of correlator receiver



$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t) \qquad i = 1,\ldots,M$$

$$\mathbf{z} = (z_1, z_2, \ldots, z_N)$$

$$z_j = \int_0^T r(t)\psi_j(t)dt \qquad j = 1,\ldots,N$$

$$N \leq M$$

White noise in orthonormal signal space

- AWGN $n(t)$ can be expressed as

$$n(t) = \underbrace{\hat{n}(t)}_{} + \underbrace{\tilde{n}(t)}_{}$$

Noise projected on the signal space which impacts the detection process.

Noise outside on the signal space

$$\begin{cases} \hat{n}(t) = \sum_{j=1}^{N} n_j \psi_j(t) \\ n_j = < n(t), \psi_j(t) > \quad j = 1, \ldots, N \\ < \tilde{n}(t), \psi_j(t) > = 0 \quad j = 1, \ldots, N \end{cases}$$

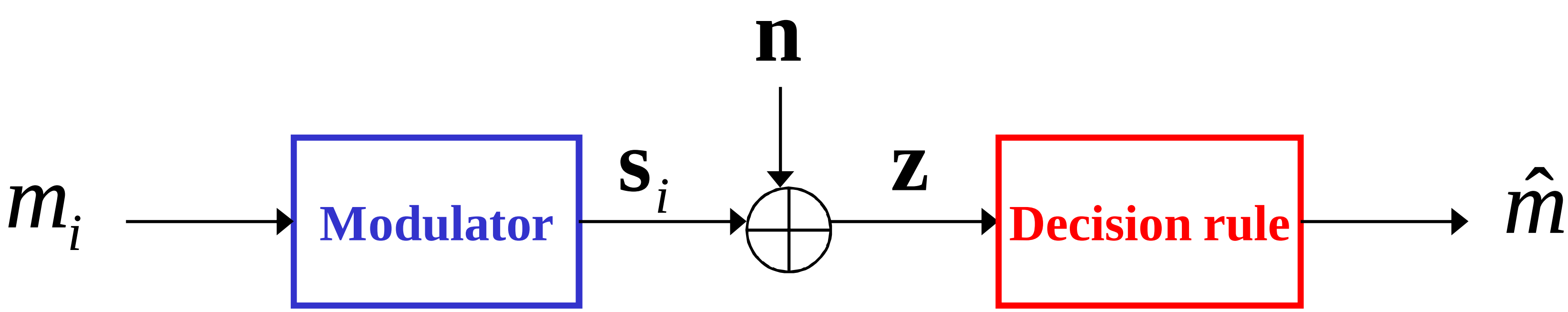
Vector representation of $\hat{n}(t)$

$$\mathbf{n} = (n_1, n_2, \ldots, n_N)$$

$\{n_j\}_{j=1}^{N}$ independent zero-mean Gaussain random variables with variance $\text{var}(n_j) = N_0 / 2$

Detection of signal in AWGN

- Detection problem:
  - Given the observation vector $\mathbf{z}$, perform a mapping from $\mathbf{z}$ to an estimate $\hat{m}$ of the transmitted symbol, m_i, such that the average probability of error in the decision is minimized.



Statistics of the observation Vector

- AWGN channel model: $\mathbf{z} = \mathbf{s}_i + \mathbf{n}$
  - Signal vector $\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})$ is deterministic.
  - Elements of noise vector $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ are i.i.d Gaussian random variables with zero-mean and variance $N_0 / 2$. The noise vector pdf is

$$p_{\mathbf{n}}(\mathbf{n}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{n}\|^2}{N_0} \right)$$

  - The elements of observed vector $\mathbf{z} = (z_1, z_2, \ldots, z_N)$ are independent Gaussian random variables. Its pdf is

$$p_{\mathbf{z}}(\mathbf{z} \mid \mathbf{s}_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{z} - \mathbf{s}_i\|^2}{N_0} \right)$$

Average probability of symbol error

- Erroneous decision: For the transmitted symbol m_i or equivalently signal vector $\mathbf{s}_i$, an error in decision occurs if the observation vector $\mathbf{z}$ does not fall inside region Z_i.
  - Probability of erroneous decision for a transmitted symbol

$$P_e(m_i) = \Pr(\hat{m} \neq m_i \text{ and } m_i \text{ sent})$$

or equivalently

$$\Pr(\hat{m} \neq m_i) = \Pr(m_i \text{ sent})\Pr(\mathbf{z} \text{ does not lie inside } Z_i \,|\, m_i \text{ sent})$$

  - Probability of correct decision for a transmitted symbol

$$\Pr(\hat{m} = m_i) = \Pr(m_i \text{ sent})\Pr(\mathbf{z} \text{ lies inside } Z_i \,|\, m_i \text{ sent})$$

$$P_c(m_i) = \Pr(\mathbf{z} \text{ lies inside } Z_i \,|\, m_i \text{ sent}) = \int_{Z_i} p_{\mathbf{z}}(\mathbf{z} \,|\, m_i)\, d\mathbf{z}$$

$$P_e(m_i) = 1 - P_c(m_i)$$

Av. prob. of symbol error …

- Average probability of symbol error :

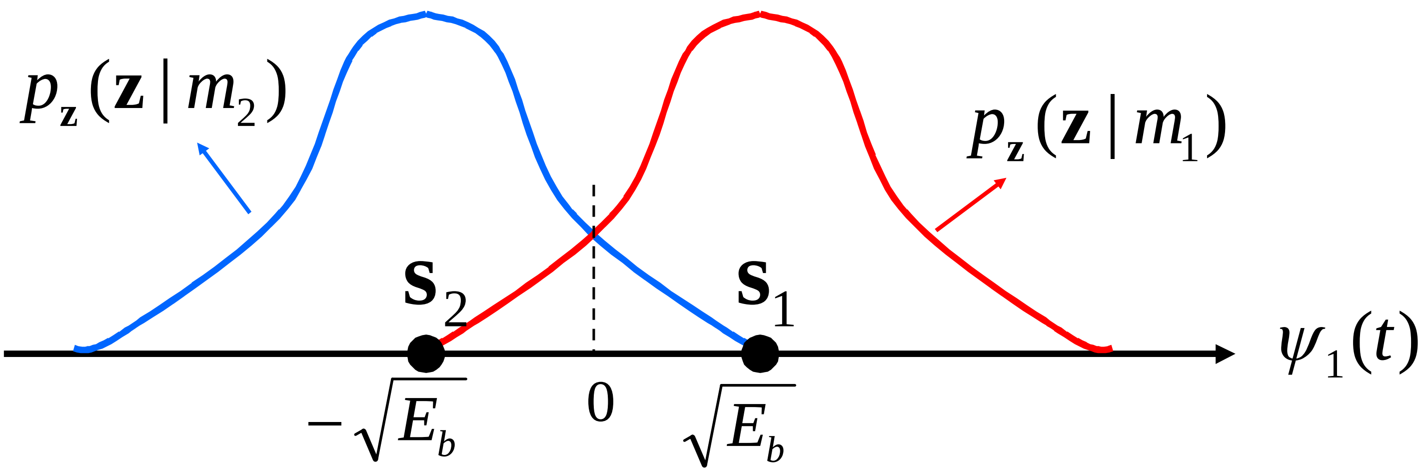
$$P_E(M) = \sum_{i=1}^{M} \Pr(\hat{m} \neq m_i)$$

  - For equally probable symbols:

$$P_E(M) = \frac{1}{M}\sum_{i=1}^{M} P_e(m_i) = 1 - \frac{1}{M}\sum_{i=1}^{M} P_c(m_i)$$

$$= 1 - \frac{1}{M}\sum_{i=1}^{M} \int_{Z_i} p_{\mathbf{z}}(\mathbf{z} \mid m_i)\, d\mathbf{z}$$

Example for binary PAM



$$P_e(m_1) = P_e(m_2) = Q\left(\frac{\|\mathbf{s}_1 - \mathbf{s}_2\| / 2}{\sqrt{N_0 / 2}}\right)$$

$$P_B = P_E(2) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Eb/No figure of merit in digital communications

- SNR or S/N is the average signal power to the average noise power. SNR should be modified in terms of bit-energy in DCS, because:
  - Signals are transmitted within a symbol duration and hence, are energy signal (zero power).
  - A merit at bit-level facilitates comparison of different DCSs transmitting different number of bits per symbol.

$$\frac{E_b}{N_0} = \frac{ST_b}{N/W} = \frac{S}{N}\frac{W}{R_b}$$

R_b : Bit rate

W : Bandwidth

Example of Symbol error prob. For PAM signals