Frequency Modulation

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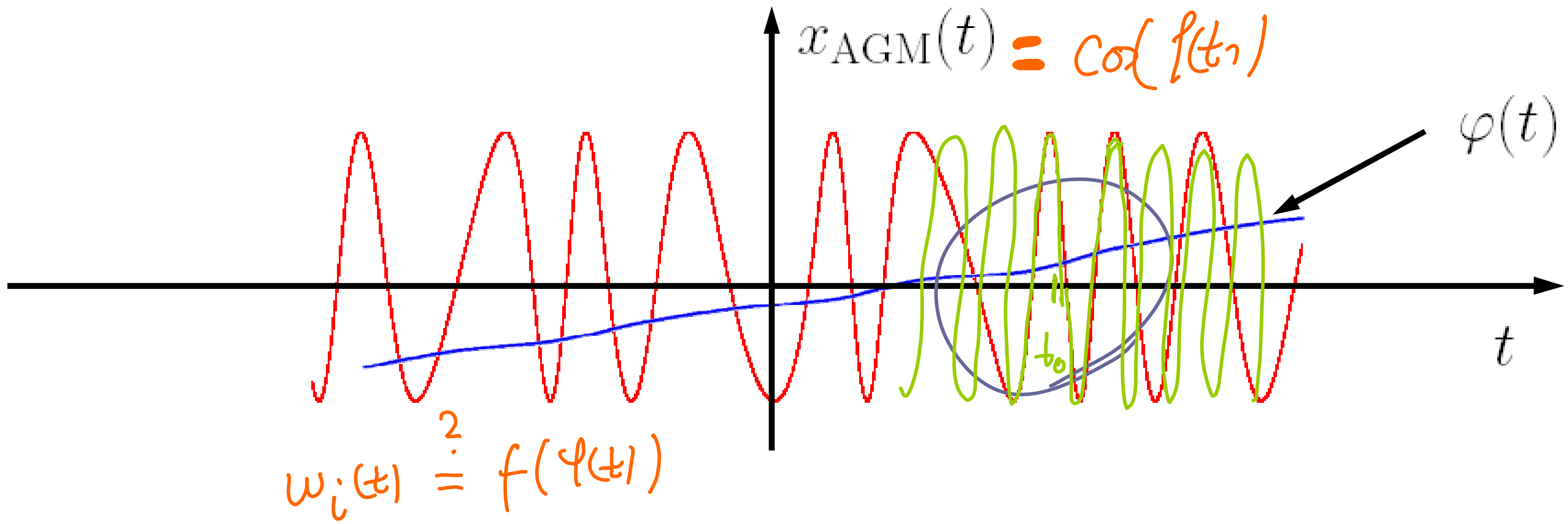
FM-signal: $$x_{\mathrm{FM}}(t) = A\cos(f_{\mathrm{FM}}(s(t))t + \phi).$$

PM-signal: $$x_{\mathrm{PM}}(t) = A\cos(\omega_c t + f_{\mathrm{PM}}(s(t))).$$

Angle Modulated signal: $$x_{\mathrm{AGM}}(t) = A\cos(\varphi(t)).$$



$$x_0(t) = A\cos(\omega_0 t + \alpha_0).$$

We require that at t_0 the signal $x_0(t)$ should coincide with $x_{\mathrm{AGM}}(t)$ in some opimal way.

$$x_0(t_0) \overset{!}{=} x_{\mathrm{AGM}}(t_0). \qquad \frac{\mathrm{d}x_0(t)}{\mathrm{d}t}\Big|_{t=t_0} \overset{!}{=} \frac{\mathrm{d}x_{\mathrm{AGM}}(t)}{\mathrm{d}t}\Big|_{t=t_0}.$$

This leads to

$$\cos(\omega_0 t_0 + \alpha_0) = \cos(\varphi(t_0)),$$

$$\omega_0 \sin(\omega_0 t_0 + \alpha_0) = \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\Big|_{t=t_0} \sin(\varphi(t_0)).$$

The first equation is fullfilled if

$$\omega_0 t_0 + \alpha_0 = \pm\varphi(t_0) + \nu 2\pi, \nu \in \mathbb{Z}.$$

Then the second equation leads to

$$\omega_0 = \pm\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\Big|_{t=t_0}.$$

Hence, it is reasonable to define the instantaneous frequency as

$$\omega_{\mathrm{i}}(t) = \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}.$$

Now we are able to specify $x_{\mathrm{PM}}(t)$ and $x_{\mathrm{FM}}(t)$.

Phase of PM-signal $\varphi(t) = \omega_c t + s(t) + \phi.$ $= f_{PM}(s(t)) + \omega_c t$

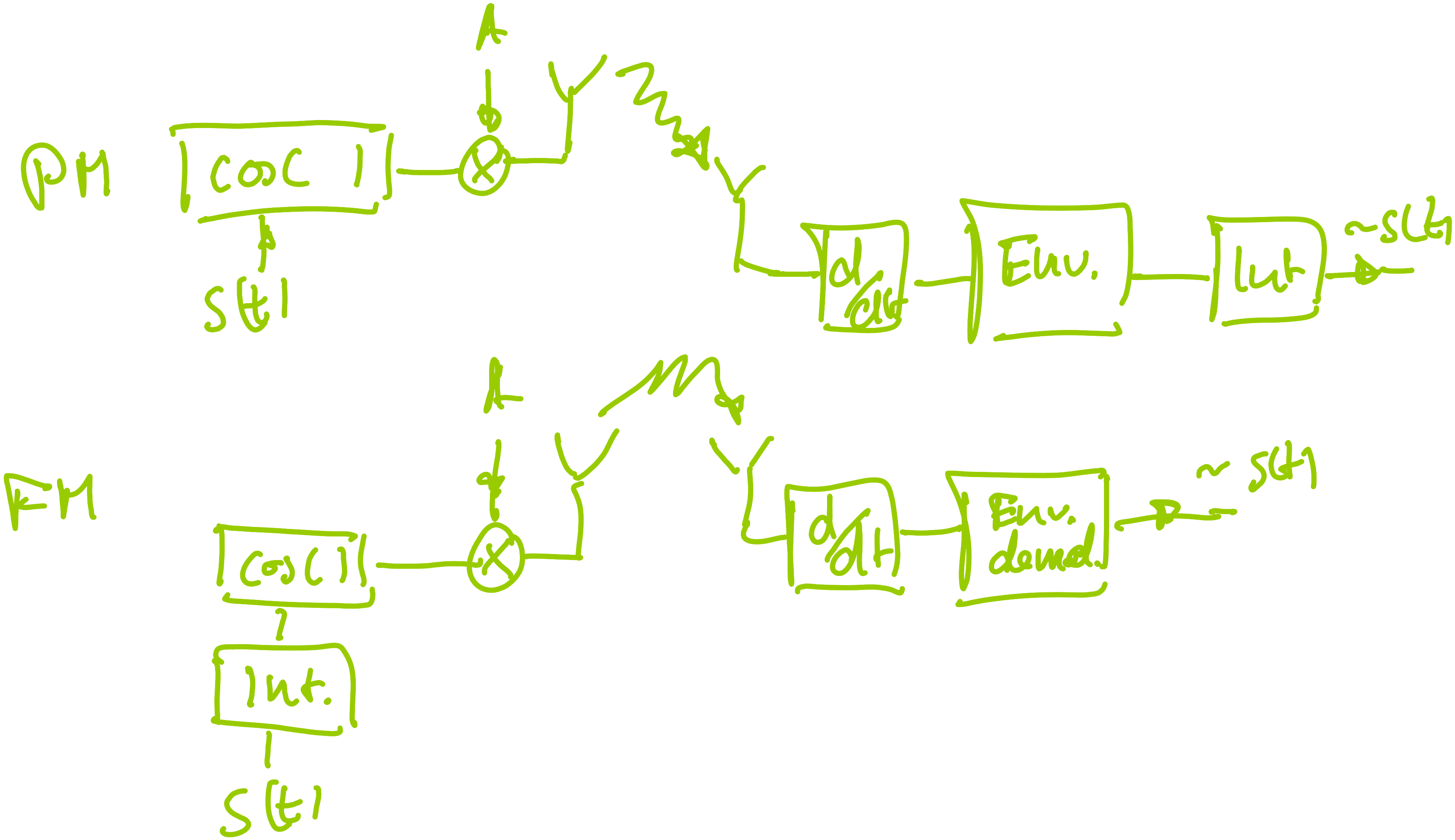
Frequency of FM-signal $\omega_{\mathrm{i}}(t) = \omega_c + s(t) = \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}.$

Consequently, we obtain $x_{\mathrm{PM}}(t) = A\cos(\omega_c t + s(t) + \phi).$

$$x_{\mathrm{FM}}(t) = A\cos(\omega_c t + \int_{-\infty}^{t} s(\tau)\mathrm{d}\tau + \phi).$$

Phase deviation $\Delta\phi_{\mathrm{FM}}$ and frequency deviation $\Delta\omega_{\mathrm{FM}}$ of FM:

$$\Delta\phi_{\mathrm{FM}} = \int_{-\infty}^{t} s(t)\mathrm{d}t, \qquad \Delta\omega_{\mathrm{FM}} = \max|s(t)|.$$


PM
cos( )
A
s(t)
d/dt
Env.
Int.
~s(t)
FM
cos( )
A
Int.
s(t)
d/dt
Env. demod.
~s(t)

Phase and frequency deviation of PM signal is also defined as

$$\Delta\phi_{\mathrm{PM}} = \max \mid s(t) \mid, \qquad \Delta\omega_{\mathrm{PM}} = \max \left|\frac{\mathrm{d}s(t)}{\mathrm{d}t}\right|.$$

- Spectrum analysis of FM-signals

$$x_{\mathrm{FM}}(t) = A\cos(\omega_c t + s_{\mathrm{I}}(t) + \phi),$$

$X_{FM}(\omega) = \int_{-\infty}^{\infty} x_{FM}(t)\, e^{-j\omega t}\, dt$

$$= A\cos(\omega_c t + \phi)\cos(s_{\mathrm{I}}(t)) - A\sin(\omega_c t + \phi)\sin(s_{\mathrm{I}}(t)),$$

where $s_{\mathrm{I}}(t) = \int_{-\infty}^{t} s(\tau)\mathrm{d}\tau.$

➢ Remember AM-signal and try to find the difference.

Let us proceed with the case $\Delta\varphi_{\mathrm{FM}} \ll 1$, which implies

$|s_{\mathrm{I}}(t)| \ll 1,\ \forall t$, since for an FM-signal $\Delta\phi_{\mathrm{FM}} = \max |s_{\mathrm{I}}(t)|$.

By use of $\cos x \overset{|x|\ll 1}{\approx} 1$ and $\sin x \overset{|x|\ll 1}{\approx} x$,

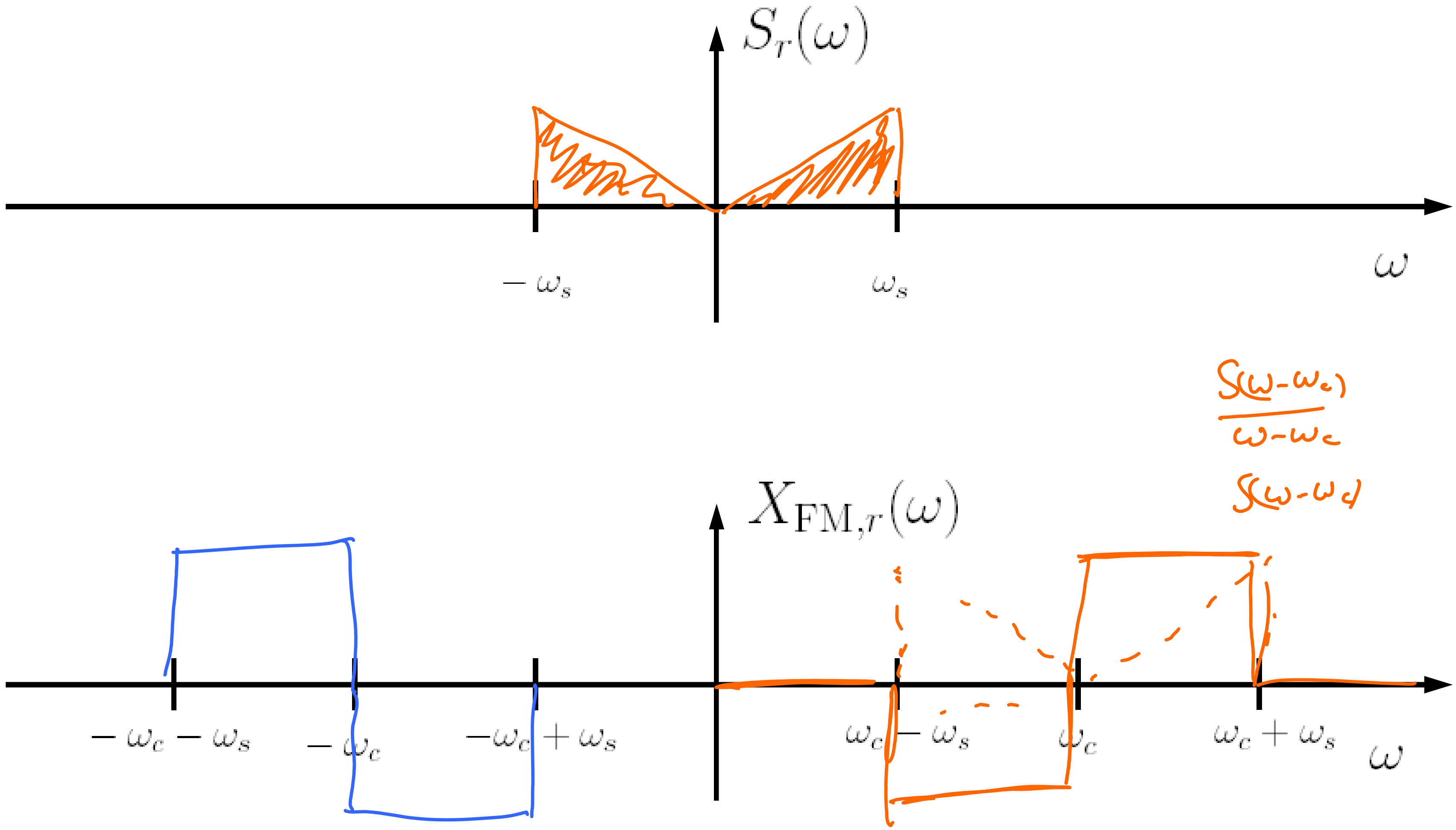
it follows $x_{\mathrm{FM}}(t)|_{\Delta\phi_{\mathrm{FM}}\ll 1} \approx A\cos(\omega_c t+\phi) - A\sin(\omega_c t+\phi)s_{\mathrm{I}}(t)$.

By use of $\int_{-\infty}^{t} s(\tau)\mathrm{d}\tau \overset{\mathcal{F}}{\circ\!\!-\!\!\bullet} \frac{1}{\mathrm{j}\omega}S(\omega)$ Fourier transform leads to

$$X_{\mathrm{FM}}(\omega)|_{\Delta\phi\ll 1} = A\pi(\delta(\omega-\omega_c)\mathrm{e}^{\mathrm{j}\phi}+\delta(\omega+\omega_c))\mathrm{e}^{-\mathrm{j}\phi} + \frac{A}{2}\frac{S(\omega-\omega_c)\mathrm{e}^{\mathrm{j}\phi}}{\omega-\omega_c} - \frac{A}{2}\frac{S(\omega+\omega_c)\mathrm{e}^{-\mathrm{j}\phi}}{\omega+\omega_c}.$$

Comparing with an AM-spectrum

$$X_{\mathrm{AM}}(\omega) = A\pi(\delta(\omega+\omega_c)\mathrm{e}^{-\mathrm{j}\phi}+\delta(\omega-\omega_c))\mathrm{e}^{\mathrm{j}\phi} + \frac{A}{2}(S(\omega+\omega_c)\mathrm{e}^{-\mathrm{j}\phi}+S(\omega-\omega_c)\mathrm{e}^{\mathrm{j}\phi}).$$



The spectrum of an FM-signal for small phase deviations

$B = 2\omega_s$

FM spectrum analysis

- In case of a small phase deviation, what is the bandwidth of an FM-signal ?

- FM is classified by „narrowband“ and „wideband“, why?

- Why the following inequality is true?

$$\omega_c + \Delta\omega_{\mathrm{FM}} \geq \omega_{\mathrm{i}}(t) \geq \omega_c - \Delta\omega_{\mathrm{FM}}.$$

FM–signal occupies a bandwidth of at least $\beta_{\mathrm{FM}} \geq 2\Delta\omega_{\mathrm{FM}}.$

Hence, we assume $\beta_{\mathrm{FM}}|_{\Delta\phi_{\mathrm{FM}} \ll 1} = 2(\Delta\omega_{\mathrm{FM}} + \alpha\omega_s).$

First rough estimation on α could be given as $\alpha \geq 1.$
Do you agree and why?

Again, we assume a sinusoidal test signal in order to calculate the FM-bandwidth

$$s(t) = \Delta\omega_{\mathrm{FM}} \cos(\omega_s t)$$

$$s_{\mathrm{I}}(t) = \frac{\Delta\omega_{\mathrm{FM}}}{\omega_s} \sin(\omega_s t) = m_{\mathrm{I}} \sin(\omega_s t), \quad m_{\mathrm{I}} = \Delta\varphi_{\mathrm{FM}} = \frac{\Delta\omega_{\mathrm{FM}}}{\omega_s}.$$

It follows

$$x_{\mathrm{FM}}(t) = A\cos(\omega_c t + s_{\mathrm{I}}(t) + \phi)$$
$$= A\cos(\omega_c t + m_{\mathrm{I}} \sin(\omega_s t) + \phi)$$

$$\Delta x_{\mathrm{FM}}(t) = A\mathrm{Re}\left\{ \mathrm{e}^{\mathrm{j}(\omega_c t + m_{\mathrm{I}} \sin(\omega_s t) + \phi)} \right\}$$
$$= A\mathrm{Re}\left\{ z(t)\mathrm{e}^{\mathrm{j}(\omega_c t + \phi)} \right\}, \quad z(t) = \mathrm{e}^{\mathrm{j}m_{\mathrm{I}} \sin(\omega_s t)}.$$

Observe that

$$z(t) \stackrel{!}{=} z(t + \nu 2\pi/\omega_s), \nu \in \mathbb{Z}$$

is periodic with period $T_s = 2\pi/\omega_s$.

Fourier series expansion yields $z(t) = \sum_{n=-\infty}^{\infty} z_n \mathrm{e}^{\mathrm{j}n\omega_s t}$

with $z_n = \frac{1}{T_s} \int_{t_0}^{t_0+T_s} z(t)\mathrm{e}^{-\mathrm{j}n\omega_s t}\mathrm{d}t, t_0 \in \mathbb{R}.$

Substituting $\omega_s t = \alpha$ and $t_0 = -T_s/2$ leads to

$$z_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} z(t)\mathrm{e}^{-\mathrm{j}(m_\mathrm{I}\sin(\alpha)-n\alpha)}\mathrm{d}\alpha$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(m_\mathrm{I}\sin(\alpha) - n\alpha)\mathrm{d}\alpha + \frac{\mathrm{j}}{2\pi} \int_{-\pi}^{\pi} \sin(m_\mathrm{I}\sin(\alpha) - n\alpha)\mathrm{d}\alpha.$$

FM signal

$$z_n = \frac{1}{\pi}\int_0^{\pi} \cos(m_\mathrm{I}\sin(\alpha) - n\alpha)\mathrm{d}\alpha \overset{!}{=} J_n(m_\mathrm{I}).$$

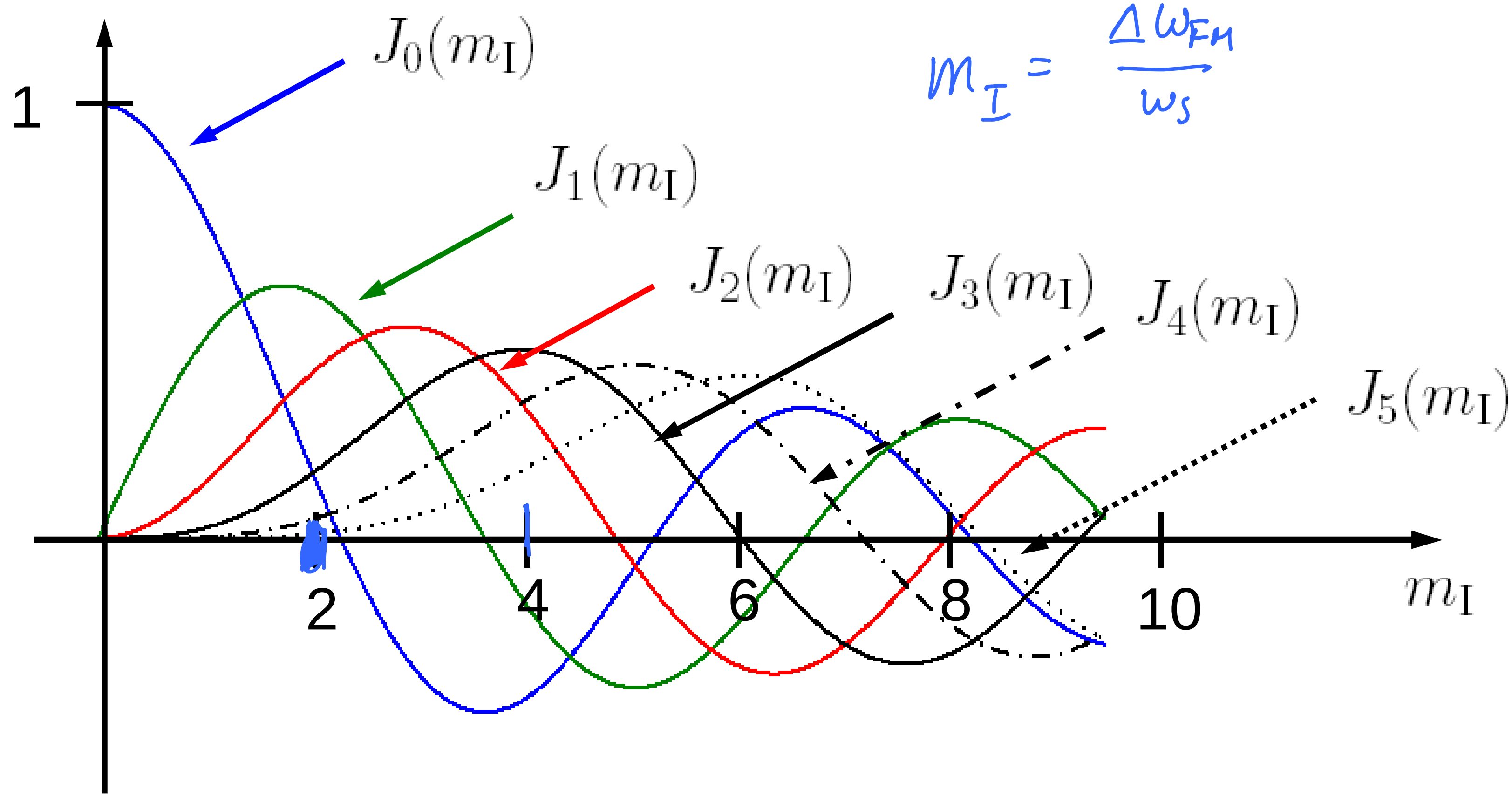
where $J_n(m_\mathrm{I})$ is the Bessel function of n-th order with property

$$\begin{aligned} J_{-n}(m_\mathrm{I}) &= \frac{1}{\pi}\int_0^{\pi} \cos(n\alpha + m_\mathrm{I}\sin\alpha)\mathrm{d}\alpha \\ &= \frac{-1}{\pi}\int_{\pi}^{0} \cos(n\pi - n\alpha + m_\mathrm{I}\sin\alpha)\mathrm{d}\alpha \\ &= (-1)^n\frac{1}{\pi}\int_0^{\pi} \cos(n\alpha - m_\mathrm{I}\sin\alpha)\mathrm{d}\alpha \\ &= (-1)^n J_n(m_\mathrm{I}). \end{aligned}$$

Bessel functions

$$J_n(m_I) = \sum_{k=0}^{\infty} \frac{(-1)^k)}{k!(n+k)!} (\frac{m_I}{2})^{n+2k}, \; n \geq 0.$$

$m_I = \frac{\Delta\omega_{FM}}{\omega_s}$



Bessel functions

$$x_{\mathrm{FM}}(t) = A\mathrm{Re}\left\{z(t)\mathrm{e}^{\mathrm{j}(\omega_c t+\phi)}\right\},$$

$$= A\mathrm{Re}\left\{\sum_{n=-\infty}^{\infty} z_n \mathrm{e}^{\mathrm{j}n\omega_s t}\mathrm{e}^{\mathrm{j}(\omega_c t+\phi)}\right\},$$

$$= A\sum_{n=-\infty}^{\infty} \mathrm{Re}\left\{J_n(m_{\mathrm{I}})\mathrm{e}^{\mathrm{j}n\omega_s t}\mathrm{e}^{\mathrm{j}(\omega_c t+\phi)}\right\},$$

$$= A\sum_{n=-\infty}^{\infty} J_n(m_{\mathrm{I}})\cos((\omega_c + n\omega_s)t + \phi).$$

$$x_{\mathrm{FM}}(t) = AJ_0(m_{\mathrm{I}})\cos(\omega_c+\phi)+$$
$$A\sum_{n=1}^{\infty} J_n(m_{\mathrm{I}})[\cos((\omega_c + n\omega_s)t + \phi) + (-1)^n\cos((\omega_c - n\omega_s)t + \phi)].$$

FM signal

$$X_{\mathrm{FM}}(\omega) = A\pi J_0(m_{\mathrm{I}})(\delta(\omega-\omega_c)\mathrm{e}^{\mathrm{j}\phi} + \delta(\omega+\omega_c)\mathrm{e}^{-\mathrm{j}\phi} +$$

$$A\pi \sum_{n=1}^{\infty} J_n(m_{\mathrm{I}})[\delta(\omega-(\omega_c+n\omega_s))\mathrm{e}^{\mathrm{j}\phi} + \delta(\omega+(\omega_c+n\omega_s))\mathrm{e}^{-\mathrm{j}\phi}$$

$$+\delta(\omega-(\omega_c-n\omega_s))\mathrm{e}^{\mathrm{j}(\phi+n\pi)} + \delta(\omega+(\omega_c-n\omega_s))\mathrm{e}^{-\mathrm{j}(\phi-n\pi)}].$$

$$x_{\mathrm{FM}}(t)|_{\Delta\varphi_{\mathrm{FM}}\ll 1} = A\cos(\omega_c t+\phi) - A\sin(\omega_c t+\phi)m_{\mathrm{I}}\sin(\omega_s t),$$

$$x_{\mathrm{FM}}(t)|_{m_{\mathrm{I}}\ll 1} = A\cos(\omega_c t+\phi) + A\frac{m_{\mathrm{I}}}{2}[\cos((\omega_c+\omega_s)t+\phi)+ \cos((\omega_c-\omega_s)t+\phi)],$$

$$\tilde{x}_{\mathrm{FM}}(t) = A\sum_{n=-N}^{N} J_n(m_{\mathrm{I}})\cos((\omega_c+n\omega_s)t+\phi) \approx x_{\mathrm{FM}}(t).$$ Why?

FM spectrum

$$\beta|_{\text{sinus}} \approx 2N\omega_s \approx 2(m_{\text{I}} + \alpha)\omega_s = 2(\Delta\omega + \alpha\omega_s),\ 1 \le \alpha \le 2.$$

$$\beta_{\text{FM,Carson}} = 2(\Delta\omega + \omega_s).$$

$$s(t) = \sum_{l=1}^{L} \Delta\omega_l \cos(\omega_{s,l} t + \phi_l),$$

$$s_{\text{I}}(t) = \sum_{l=1}^{L} m_{\text{I},l} \sin(\omega_{s,l} t + \phi_l),\ m_{\text{I},l} = \frac{\Delta\omega_l}{\omega_{s,l}}.$$ Why?

$$z(t) = \prod_{l=1}^{L} z_l(t),\ z_l(t) = \mathrm{e}^{\mathrm{j}(m_{\text{I},l} \sin(\omega_{s,l} t) + \phi_l)}.$$

$$z_l(t) = \sum_{n_l=-\infty}^{\infty} J_{nl}(m_{\text{I},l}) \mathrm{e}^{\mathrm{j}n_l(\omega_{s,l} t + \phi_l)}.$$

FM spectrum

Now, we find spectral lines at $\omega_c + \sum_{n_l=-\infty}^{L} n_l \, \omega_{s,l}$ with amplitudes $A \prod_{l=1}^{L} J_{nl}(m_{\mathrm{I},l})$.

- Demodulation of FM

$$x_{\mathrm{FM}}(t) = A\cos(\omega_c t + s_{\mathrm{I}}(t) + \phi).$$

$$s(t) = \frac{\mathrm{d}s_{\mathrm{I}}(t)}{\mathrm{d}t}, \qquad y_{\mathrm{FM}}(t) = x_{\mathrm{FM}}(t).$$

$$\frac{\mathrm{d}y_{\mathrm{FM}}(t)}{\mathrm{d}t} = A\frac{\mathrm{d}(\omega_c t + s_{\mathrm{I}}(t))}{\mathrm{d}t}\sin(\omega_c t + s_{\mathrm{I}}(t) + \phi)$$

$$\stackrel{!}{=} A(\omega_c + s(t))\sin(\omega_c t + s_{\mathrm{I}}(t) + \phi).$$

Can we stop modulation here?

$$H(\omega) = \begin{cases} \mathrm{j}\omega & \text{for} \quad \omega_c + \beta_{\mathrm{FM}}/2 \geq |\omega| \geq \omega_c - \beta_{\mathrm{FM}}/2. \\ \text{arbitrary} & \text{else} \end{cases}$$

- In broadcast radios FM is the most popular analog modulation technique. Do you agree? Discuss.

- Envelope detection is possible in AM as well as FM. Comment.

- Envelope detection shows better performance in FM than AM. Do you agree, why?

- AM bandwidth is not adjustable as FM band width. Give two reasons.

- For better performance, FM adjusts the modulation index instead of the transmitted power as in case of AM. Do you agree? If yes, which trade-off could we face?