Self-dualized Threshold Functions

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Abstract
Threshold functions are Boolean functions that model neurons, processing units of an artificial neural network. The enumeration of threshold functions of five or fewer variables, generated by a computer, given in [10] exhibits an interesting phenomenon: the number of (n, 2^{n-1}) threshold functions is the same as the number of (n-1)-variable threshold functions. This paper shows that every (n, 2^{n-1}) threshold function is self-dual and there is a one-to-one correspondence between the set of all (n, 2^{n-1}) threshold functions and the set of all (n-1)-variable threshold functions. An algorithm for generating threshold functions by self-dualization on threshold functions is given.

Key words: threshold function, minterm, self-dual

1. Introduction

A Boolean function f of n variables $X = (x_1, x_2, \ldots, x_n)$ is a *threshold function* if there exists a set of real numbers $w_1, w_2, \ldots, w_n$, called (*input*) *weights,* and θ, called the *threshold*, such that the following conditions are satisfied:

$$f(X) = 1 \quad \text{if} \sum_{1}^{n} x_i w_i \geq \theta$$
$$f(X) = 0 \quad \text{otherwise.} \tag{1}$$

In this case, $[w_1, w_2, \ldots, w_n; \theta]$ is called a structure of f.

The logical product, $f = x_1 x_2$ ($x_1 AND x_2$) is a threshold function that can be realized by [1, 1; 2] or [1, 2; 3]. In general, a threshold function has an infinite number of structures. The logical summation $g = x_1 + x_2$ ($x_1 OR x_2$) is another threshold function. One of its structures is [1, 1; 1]. However, $x_1 \overline{x_2} + \overline{x_1} x_2$ ($x_1 XOR x_2$) is not a threshold function.

In this paper, each Boolean function is represented in the *minterm expansion form* (mef). The minterm expansion form is a disjunction of different minterms. A *minterm* is a conjunction of different *literals*, variables or their complements, in which each variable is involved exactly once. Every Boolean function f can be represented uniquely in the mef. For instance, the mef of $f = x_1 + \overline{x_2} x_3$ is $f = x_1 x_2 x_3 + x_1 x_2 \overline{x_3} + x_1 \overline{x_2} x_3 + x_1 \overline{x_2}\,\overline{x_3} + \overline{x_1}\,\overline{x_2} x_3 + \overline{x_1}\,\overline{x_2} x_3$.

If the number of variables is $n,$ then there are 2^n different minterms. A minterm is called a *true* minterm (or a minterm) of f if it is appeared in the mef of f. Otherwise it is called a *false* minterm.

For simplicity, $S(f)$ represents the set of all *true* minterms of f and D represents the set of all minterms of n variables. A Boolean function of n variables having k minterms is called an (n, k) Boolean-function. If it is a threshold function, it is called an (n, k) threshold-function.

Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be *true* minterms of f. If x_i appears uncomplemented (or complemented) in both $\mathbf{m}_1$ and $\mathbf{m}_2$, we say that $\mathbf{m}_1$ and $\mathbf{m}_2$ are identical in the i-th literal. Suppose that $\mathbf{m}_1$ and $\mathbf{m}_2$ are identical in k out of n literals. The (logical) product of these common literals form a term **s** called the identical subminterm between $\mathbf{m}_1$ and $\mathbf{m}_2$. The $(n\text{-}k)$ complementary literals in $\mathbf{m}_1$ and $\mathbf{m}_2$ form two complementary terms **d** and **~d** called the differing subminterms between $\mathbf{m}_1$ and $\mathbf{m}_2$. Hence, $\mathbf{m}_1$ and $\mathbf{m}_2$ can be written as **sd** and **s~d** respectively. Negating a minterm means negating each literal of the minterm.

The enumeration of threshold functions having up to five variables, generated by computers, as given in [10], shows that the number of $(n, 2^{n-1})$ threshold-functions is the same as the number of $(n\text{-}1)$-variable threshold functions. To find out whether or not this fact holds for any number of variables, we attempted to generate $(n, 2^{n-1})$ threshold-functions from $(n\text{-}1)$-variable threshold functions by self-dualization. Next, we induce a 1-assignment to an $(n, 2^{n-1})$ threshold-function to generate an $(n\text{-}1)$-variable threshold function.

Since self-dualization of an $(n\text{-}1)$-variable threshold function always results in an $(n, 2^{n-1})$ threshold-function, an algorithm for generating threshold functions by self-dualization is developed. The algorithm produces all $(n, 2^{n-1})$ threshold functions if all $(n\text{-}1)$-variable threshold functions are given as the inputs. Although the algorithm does not produce all n-variable threshold functions, it works faster since it does not use a lot of comparison like the generating algorithms given in [10].

2. Some Properties of Threshold Functions

2.1. Preserving Operations and Closure Transformations

If f is a threshold function of n variables $x_1, x_2, \ldots, x_n$, then the (logical) addition and multiplication between f and a variable are also threshold functions. In other words,

(1) $f + x_p$,

(2) fx_p

are threshold functions, where $1 \le p \le n+1$ (see [8]).

Besides the closure transformations of threshold functions there are three preserving operations as follows. Given a threshold function f, the Boolean function g that can be obtained from f by one or combinations of the following operations is a threshold function, as given in [8]:

(1) negation of one or more variables;
(2) permutation of two or more variables; and
(3) negation of the output function, i.e., $\overline{f}$.

2.2. Complete Monotonicity

Consider an n-variable threshold function f. It is called completely monotonic if every pair of *true* minterms $\mathbf{m}_i$ and $\mathbf{m}_j$ the following condition holds: if **s** is the identical subminterm between $\mathbf{m}_i$ and $\mathbf{m}_j$, for every other pair of minterms that have **s** as their identical subminterm then at least one of the two is a *true* minterm of *f*. For example, $f=\overline{x_1}x_2x_3+x_2\overline{x_2}x_3$ is not completely monotonic. The identical subminterm of $\overline{x_1}x_2x_3$ and $x_2\overline{x_2}x_3$ is x_3. Consider other pair of minterms that are identical in x_3: $\overline{x_1}\,\overline{x_2}x_3$ and $x_1x_2x_3$. Both of them are *false* minterms of *f*.

Every threshold function is completely monotonic and every completely monotonic function of up to eight variables is a threshold function, see [16].

3. Self-dualization and Assignment

Given an *n*-variable threshold function *f*, the self-dualized function of *f* with respect to a new variable x_{n+1} is defined as

$$f^H = fx_{n+1} + f^d\overline{x}_{n+1} \tag{2}$$

where f^d, the dual function of *f*, is obtained from *f* by negating all the variables and followed by negating the output:

$$f^d(X) = \overline{f}(\overline{X}). \tag{3}$$

In terms of minterms, $\mathbf{m}_i$ is a minterm of *f*(*X*) if and only if ~$\mathbf{m}_i$, the negation of $\mathbf{m}_i$, is a minterm of $f(\overline{X})$, written as $\mathbf{m}_i \in S(f(X))$ if and only if $\sim\mathbf{m}_i \in S(f(\overline{X}))$. The set $S(\overline{f}(X))$ consists of all *false* minterms of *f*(*X*). Thus, $S(f^d) = S(\overline{f}(\overline{X}))$ = D - $S(f(\overline{X}))$. Note that $S(f(X))$ and $S(f(\overline{X}))$ do not have to be disjoint.

A Boolean function is said to be *self- dual* if and only if $f=f^d$. Thus, *f* is self-dual implies $\mathbf{m}_i$ is a *true* minterm of $\overline{f}$ if and only if ~$\mathbf{m}_i$ is a *true* minterm of *f*. As shown in [16] that f^H, named as the hyper function of *f*, is self-dual and *f* is a threshold function if and only if f^H is.

Given an (*n*-1, *k*) threshold function *f*. The number of $\overline{f}$'s minterms is the same as the number of all possible minterms subtracted by the number of *f* 's minterms, 2^{n-1}- *k*. Since $\overline{f}$ and f^d have the same number of minterms, then the number of f^H's minterms is $k + 2^{n-1} - k = 2^{n-1}$. Therefore, self-dualization of an (*n*-1)-variable threshold function always results in an (*n*, 2^{n-1}) threshold-function.

Let *f* an (*n*, *k*) self-dual threshold function, $f=f^d$. Thus, f^d contains *k* minterms also. On the other hand, $\overline{f}$ and f^d have the same number of minterms, i.e., 2^n- *k*. Therefore, every *n*-variable self-dual threshold function has $k = 2^{n-1}$ minterms. In other words, *n*-variable threshold functions having *k* minterms are not self-dual if $k \neq 2^{n-1}$. The following theorem shows that every (*n*, 2^{n-1}) threshold-function is self-dual.

Theorem 1:
Every (*n*, 2^{n-1}) threshold-function is self-dual.

Proof:
Let *f* be an (*n*, 2^{n-1}) threshold-function. Based on complete-monotonicity property of threshold function, for every *true* minterm **m** of *f*, its complement, ~**m**, is a *false* minterm of *f*. Otherwise,

every pair of complementary minterms are *true* minterms; and hence the number of *true* minterms of f is 2^n. This contradicts to the fact that the number of *true* minterms of f is 2^{n-1}. Similarly, if **m** is a *false* minterm of f, then ~ **m** is a true minterm of f, written as

$$(\mathbf{m} \in \mathrm{S}(f)) \Leftrightarrow (\sim \mathbf{m} \in \mathrm{S}(\overline{f})). \quad (4)$$

Note that

$$(\sim\mathbf{m} \in \mathrm{S}(\overline{f})) \Leftrightarrow (\mathbf{m} \in \mathrm{S}(f^d)). \quad (5)$$

From (4) and (5)

$$(\mathbf{m} \in \mathrm{S}(f)) \Leftrightarrow (\mathbf{m} \in \mathrm{S}(f^d)). \quad (6)$$

We conclude that f is self-dual, i.e., $f = f^d$.

Let f_1 and f_2 be two different threshold functions of (n-1) variables $x_1, x_2, \ldots, x_{n-1}$ and x_n be a new variable for both f_1 and f_2. Since f_1 and f_2 are different, there is a minterm **m** of f_1 that is not a minterm of f_2; hence, $\mathbf{m}x_n$ is a minterm of f_1^H but not a minterm of f_2^H. Consequently, $f_1^H \neq f_2^H$. In other words, $f_2 \neq f_2$ implies $f_1^H \neq f_2^H$. Thus, self-dualization can be regarded as a one-to-one mapping from the set of all (n-1)-variable threshold functions to the set of all (n, 2^{n-1}) threshold-functions. The next question is whether or not the mapping is surjective. To answer the question, we investigate how to generate an (n-1)-variable threshold function from an (n, 2^{n-1}) threshold-function using 1-assignment.

Let $X = (x_1, x_2, \ldots\ x_n)$. A_j is a k-assignment on X if A is a mapping from a subset having k elements X_A of the variables into {0, 1} and f_A represents a function obtained from f induced by assignment A. In particular, A is a 1-assignment if A is assignment $\{x_j = 1\}$ or $\{x_j = 0\}$, $1 \le j \le n$.

Let A be assignment $\{x_j = 1\}$. Let's consider all two variables threshold functions having two minterms. Assigning one of its variables to either 1 or 0 gives all threshold functions having one variable, as shown by the following table. On the other hand, self-dualization will convert the functions back to the original functions.

Table 1: (2, 2) Threshold Functions Induced by 1-Assignment and Self-dualization

No.	(2, 2) threshold-function f	f^d	f_A	$f_A^d = f_{\overline{A}}$	$f_A^H = f_A x_2 + f_A^d \overline{x}_2$
1	$x_1\overline{x}_2 + \overline{x}_1\overline{x}_2$	$x_1\overline{x}_2 + \overline{x}_1\overline{x}_2$	0	$x_1 + \overline{x}_1$	$x_1\overline{x}_2 + \overline{x}_1\overline{x}_2$
2	$x_1x_2 + \overline{x}_1x_2$	$x_1x_2 + \overline{x}_1x_2$	$x_1 + \overline{x}_1 = 1$	0	$x_1x_2 + \overline{x}_1x_2$
3	$x_1x_2 + x_1\overline{x}_2$	$x_1x_2 + x_1\overline{x}_2$	x_1	$\overline{x}_1$	$x_1x_2 + x_1\overline{x}_2$
4	$x_1x_2 + \overline{x}_1\overline{x}_2$	$x_1x_2 + \overline{x}_1\overline{x}_2$	$\overline{x}_1$	x_1	$x_1x_2 + \overline{x}_1\overline{x}_2$

The last column shows that every (2, 2) threshold-function f can be decomposed into two threshold functions $f_1 = f_A$ and $f_2 = f_A^d = f_{\overline{A}}$ such that $f = f_A^H = f_A x_2 + f_A^d \overline{x}_2$, where A is a 1-assignment. Does this principle apply to any number of variables? In other words, does 1-assignment on an n-variable threshold function always result in an (n-1)-variable threshold function?

Given a threshold function f of n variables $x_1, x_2, \ldots, x_n$. We gather all minterms in which x_n appears uncomplemented in one group and those in which x_n appears complemented into another group. Then f can be expressed as $f = f_1 x_n + f_2 \overline{x_n}$. We induce f by 1-assignment $A = \{x_n = 1\}$ to get f_1, $f_1 = f_A$. Similarly, we construct f_2 by inducing f by $\overline{A}$, $f_2 = f_{\overline{A}}$. The following lemma shows that if f is a threshold function then f_A is threshold function and so is $f_{\overline{A}}$.

Lemma:
Every threshold function f of n variables $x_1, x_2, \ldots, x_n$ can be decomposed into two threshold functions f_A and $f_{\overline{A}}$ such that

$$f = f_A x_n + f_{\overline{A}} \overline{x_n} \tag{7}$$

where f_A and $f_{\overline{A}}$ are threshold functions of (n-1) variables $x_1, x_2, \ldots, x_{n-1}$.

Proof:
Every Boolean function f can be decomposed into f_A and $f_{\overline{A}}$ such that $f = f_A x_n + f_{\overline{A}} \overline{x_n}$, where $A = \{x_n = 1\}$. Since f is a threshold function, then there exists a structure $[w_1, w_2, \ldots, w_n; \theta]$ that realizes f. This means,

$$(\sum_1^n x_i w_i \geq \theta) \Rightarrow f(X) = 1 \tag{8}$$

and

$$(\sum_1^n x_i w_i < \theta) \Rightarrow f(X) = 0 \tag{9}$$

Since $x_n = 1$ then (8) and (9) can be written as

$$(\sum_1^{n-1} x_i w_i \geq \theta - w_n) \Rightarrow f(X) = 1 \tag{10}$$

and

$$(\sum_1^{n-1} x_i w_i < \theta - w_n) \Rightarrow f(X) = 0 \tag{11}$$

Note that if $f = 0$ then $f_A = 0$; and if $f = 1$ then f_A. Consequently,

$$(\sum_1^{n-1} x_i w_i \geq \theta - w_n) \Rightarrow f_A(X) = 1 \tag{12}$$

and

$$(\sum_1^{n-1} x_i w_i < \theta - w_n) \Rightarrow f_A(X) = 0 \tag{13}$$

These prove that f_A is realizable by $[w_1, w_2, ..., w_{n-1}; \theta - w_n]$. Similarly, since $\overline{A} = \{x_n = 0\}$ then $f_{\overline{A}}$ is realizable by $[w_1, w_2, ..., w_{n-1}; \theta]$. Thus, 1-assignment on a threshold function always results in a threshold function having one fewer variable. In particular, if $f_{\overline{A}} = f_A^d$, the decomposition of f given in (7) is a self-dualization of f_A. This condition is satisfied when f is self-dual or f is an (n, 2^{n-1}) threshold-function, as given by the following theorem.

Theorem 2:
Every (n, 2^{n-1}) threshold-function can be obtained by self-dualization of an (n-1)-variable threshold function g

$$f = gx_n + g^d \overline{x}_n \tag{14}$$

Proof:
Let f be an (n, 2^{n-1}) threshold-function. Consider the decomposition of f and f^d, $f = f_A x_n + f_{\overline{A}} \overline{x}_n$ and $f^d = f_A^d x_n + f_{\overline{A}}^d \overline{x_n}$. Since f is self-dual, then $f = f_A x_n + f_{\overline{A}} \overline{x}_n = f^d = f_A^d x_n + f_{\overline{A}}^d \overline{x_n}$. Therefore, f_A and $f_{\overline{A}}$ are self-dual threshold functions and hence each of them has 2^{n-2} minterms. The only possibility is that $f_A = f_{\overline{A}}$ ($f_A = f_{\overline{A}} = g = g^d$).

Suppose that $f_A \neq f_{\overline{A}}$. Since they have the same number of minterms, 2^{n-2}, then there are minterms $\mathbf{m_1} \in S(f_A)$, $\mathbf{m_1} \notin S(f_{\overline{A}})$ and $\mathbf{m_2} \in S(f_{\overline{A}})$, $\mathbf{m_2} \notin S(f_A)$. Since $f = f_A x_n + f_{\overline{A}} \overline{x}_n$, then $\mathbf{m_1} x_n \in S(f)$, $\mathbf{m_2} x_n \notin S(f)$, $\mathbf{m_1} \overline{x_n} \notin S(f)$, $\mathbf{m_2} \overline{x_n} \in S(f)$. As a result, f is not completely monotonic and therefore it is not a threshold function. It is a contradiction.

4. Algorithm

The program given in [10] generates threshold functions of up to five variables. For threshold functions having six variables, the program becomes incredibly slow since it selects all completely monotonic Boolean functions from huge number of generated functions. The following algorithm generates all (n, 2^{n-1}) threshold functions using self-dualization in a faster way (without checking the complete monotonicity property) with the set of all (n-1) threshold functions as the inputs.

In implementing the algorithm into a computer program, we need to represent minterms and functions. For instance, a function can be represented as a set of non-negative integers. Each integer m represents the decimal version of the binary representation b of the minterm **m**. If the number of variables is n, the binary representation of a mintem **m** is a binary number b of length n; if x_i appeared uncomplemented in **m** then the i-th digit of b is 1, and 0 otherwise.

If S(f) is the set of f 's minterms, then S($\overline{f}$) = D - S(f), where D is the set of all possible minterms of n variables. The negation of minterm **m**, represented by m, is a minterm represented by $2^n - m$. Multiplying f by x_n can be done by multiplying each integer in S(f) by 2 followed by adding the result by 1. Multiplying each integer in S(f) by 2 will produce $f\overline{x_n}$. The following algorithm generates (n, 2^{n-1}) threshold functions by self-dualization, where threshold functions are represented as sets of integers in D.